The ESSENTIALS of ELECTRIC CIRCUITS I

Staff of Research and Education Association,
Dr. M. Fogiel, Director

This book covers the usual course outline of Electric Circuits I. For more advanced topics, see *"THE ESSENTIALS OF ELECTRIC CIRCUITS II"*.

REA
Research and Education Association
61 Ethel Road West
Piscataway, New Jersey 08854

WHAT "THE ESSENTIALS" WILL DO FOR YOU

This book is a review and study guide. It is comprehensive and it is concise.

It helps in preparing for exams, in doing homework, and remains a handy reference source at all times.

It condenses the vast amount of detail characteristic of the subject matter and summarizes the **essentials** of the field.

It will thus save hours of study and preparation time.

The book provides quick access to the important facts, principles, theorems, concepts, and equations of the field.

Materials needed for exams, can be reviewed in summary form – eliminating the need to read and re-read many pages of textbook and class notes. The summaries will even tend to bring detail to mind that had been previously read or noted.

This "ESSENTIALS" book has been carefully prepared by educators and professionals and was subsequently reviewed by another group of editors to assure accuracy and maximum usefulness.

Dr. Max Fogiel
Program Director

CONTENTS

CHAPTER 1

UNITS AND NOTATION

1.1 SYSTEMS OF UNITS

1.1.1 THE SI SYSTEM

The International System of Units is based on six basic units: 1) meter, 2) kilogram, 3) second, 4) ampere, 5) degree kelvin, and 6) candela.

1.1.2 STANDARD ABBREVIATIONS FOR SI

A - ampere
ac - alternating current
C - coulomb
cps - cycle per second
dc - direct current
dB - decibel
eV - electron volt
F - farad
ft - foot
g - gram
H - henry
h - hour
Hz - hertz
J - joule
kg - kilogram
m - meter
min - minute
mks - meter-kilogram-second
N - newton
N•m - newton-meter
Np - neper
PF - power factor
rad - radian
RLC - resistance-inductance-capacitance
rms - root-mean-square
rps - revolutions per second
s - second
V - volt
VA - voltampere
W - watt
Wh - watthour
^{0}F - degree Farenheit
^{0}C - degree celsius
^{0}K - degree kelvin
Ω - ohm
℧ - mho

1.1.3 THE MKS AND CGS SYSTEM (METRIC SYSTEM)

	(m,kg,sec) MKS	(cm,g,sec) CGS
1. Length (ℓ)	meter	centimeter (cm)
2. Mass (m)	kilogram	gram (g)
3. Time (t)	second	second
4. Force (F or f) *Note: F[N] = m[kg] × Acceleration (a) [m/s^2]	newton	Dyne
5. Work and Energy (W or w)	newton-meter (joule)	Dyne-centimeter (or Erg)
6. Power (P or p)	joule/sec (watt)	

1.1.4 THE ENGLISH SYSTEM

1. Length → yard (yd)
2. Mass → slug
3. Time → second
4. Force → Pound (lb)
5. Energy → foot-pound (ft-lb)

1.1.5 UNITS CONVERSION BETWEEN ENGLISH MKS AND CGS SYSTEMS

	English	MKS	CGS
Length:	1 yd = 0.914m 1 in = 0.0254m	1m = 39.37 in = 100 cm	2.54 cm = 1 in
Mass:	1 slug = 14.6 kg	1 kg = 1000 g 0.45359237 kg = 1 lbm	———
Force:	1 lb = 4.45 N	1N = 0.22481 lb_f 1N = 100,000 dynes	———

	English	MKS	CGS
Energy:	1 ft-lb = 1.356 J	1J = 0.7376 ft-lb=1N·m	$1J=10^7$ ergs
Power:	————	$1J/s = 0.7376$ ft-lb_f/s = 11745.7 hp	————
Temperature:	$^0F = \frac{9}{5} C^0 + 32$	$^0C = \frac{5}{9} (F^0 - 32)$	————

1.2 LAWS OF UNITS

1. In each term of an equation, the units of measurement must be the same.

2. Only one system of units is used with any one equation; both sides of the equality must be of the same system.

1.3 UNIT OF CHARGE AND COULOMB'S LAW

Charge: Symbol → Q - constant charge
q - time-varying or instantaneous value of charge, i.e., g(t).

Unit → Coulomb (C). (Note: the charge of an electron = -1.60219×10^{-19} C, where 1 C (negative) = combined charge of about 6.24×10^{18} electrons. The charge of one proton = $+1.602 \times 10^{-19}$ C.)

Coulomb's Law:

$$\boxed{F = \frac{k\, Q_1 Q_2}{d^2} = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{d^2} \text{ [N]}}$$

where $k = 9 \times 10^9 \, N\text{-}m^2/C^2$ = proportionality constant

$= (4\pi\varepsilon_0)^{-1}$

$Q_1 + Q_2$ = charge of 2 bodies in coulomb

d = separated distance between 2 charged bodies

and ε_0 = permittivity of free space = $8.85 \times 10^{-12} \, C^2/N \cdot m^2$

Note: 1) For any material: ε = permittivity of the material = $K\varepsilon_0$, where K = dielectric constant

2) In a vaccum: K = 1 and $\varepsilon = \varepsilon_0$.

I.4 SCIENTIFIC NOTATION

Power of Ten	Prefix	Abbreviation
10^{-18}	atto	a
10^{-15}	femto	f
* 10^{-12}	pico	p
* 10^{-9}	nano	n
* 10^{-6}	micro	μ
* 10^{-3}	milli	m
* 10^{-2}	centi	c
* 10^{-1}	deci	d
10^{1}	deka	da
10^{2}	hecto	h
* 10^{3}	kilo	k
* 10^{6}	mega	M
* 10^{9}	giga	G
10^{12}	tera	T

(* - most frequently used)

CHAPTER 2

RESISTIVE CIRCUITS AND EXPERIMENTAL LAWS

2.1 CURRENT, VOLTAGE, POWER AND ENERGY

2.1.1 CURRENT

Definition:

The measurement of the rate of the number of charges moving through a given reference point in a circuit in 1 second. For steady current,

$$\boxed{i = \frac{q}{t}},$$

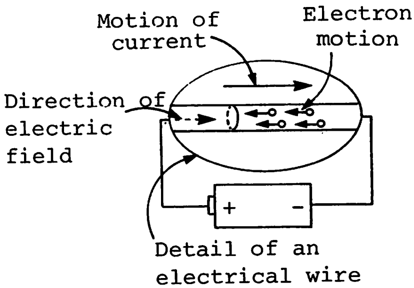
where q is the net charge passing through the point in t seconds.

Unit of current = ampere (A) = 1 coulomb of charge moving past a point in 1 second.

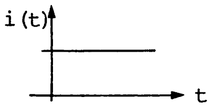
Instantaneous current (i) = time rate of change of charge = $\frac{dq}{dt}$.

Current Flow: The current flow in a wire is opposite to the motion of the electrons by convention. (See Fig.)

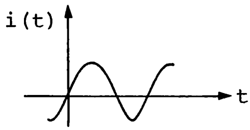
Note: Current flow in the opposite direction of the figure is given a negative sign.



Direct Current (DC): A current which is constant due to a steady, unchanging, unidirectional flow of charge.



Alternating Current (AC): Sinusoidal time varying current, e.g., household current.



2.1.2 VOLTAGE

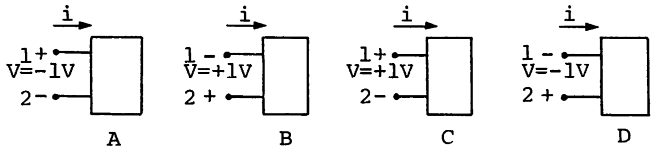
Definition:

The voltage (V or v), or the potential difference between two points, is the measure of the work required to move a unit charge from one point to another.

Unit of voltage = volt = 1 joule/coulomb

Voltage Sign Convention:

Assume a positive current supplied by an external source is entering terminal 1. Then,

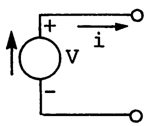


Terminal 1 is 1 volt positive with respect to terminal 2 - Figs. C & D and Terminal 2 is 1 volt positive with respect to terminal 1 - Figs. A & B.

2.1.3 POWER AND ENERGY

Definition:

(p)power [watts] = v[volts] $\times$ i[amperes]



Efficiency: $\eta = \dfrac{\text{power output}}{\text{power input}}$ $\qquad 0 < \eta < 1$

Energy:

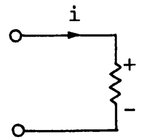
Since power (p) is the time rate of energy transfer $(p = \frac{dW}{dt})$

$$\boxed{W = \int_{t_1}^{t_2} p\,dt}$$

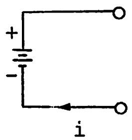
(the energy transferred during a given time interval x) or, W(Energy in watt-seconds or joules) = p(power in watts) $\times$ t(time in seconds).

Energy and Voltage Relationships:

1. voltage drop across element $\rightarrow$ positive released energy



2. voltage rise $\longrightarrow$ positive generated energy



2.2 TYPES OF CIRCUIT ELEMENTS

2.2.1 INDEPENDENT VOLTAGE SOURCE

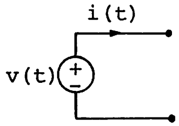
Characteristics:

1. The voltage between the two terminals is independent of the current through it.

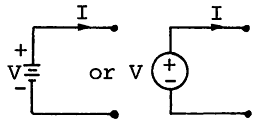
2. The same amount of voltage output is supplied continuously regardless of the amount of current drawn from it.

Types:

A) time-varying



B) time-invarying (independent DC voltage source)(i.e., constant terminal voltage).

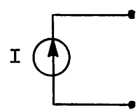


2.2.2 INDEPENDENT CURRENT SOURCE

Characteristic:

The current supplied by the source is fixed to a load and is completely independent of the voltage across it.

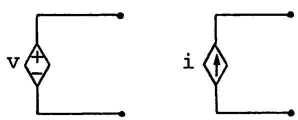
Circuit symbol:



Note: Both independent current and voltage sources are approximations for a physical element.

2.2.3 DEPENDENT VOLTAGE AND CURRENT SOURCES

Circuit symbol:



(a) dependent voltage source (b) dependent current source

Characteristic:

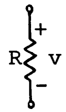
The source quantity of a dependent source is determined by a voltage or current existing at some other location in the electrical system under consideration.

2.3 RESISTANCE AND CONDUCTANCE

2.3.1 RESISTANCE

The measure of the tendency of a material to impede the flow of electric charges through it.

Circuit symbol:



R = resistance of the resistor having units [volts/ampere] or ohm (Ω).

2.3.2 CONDUCTANCE

The reciprocal of resistance, or the ratio of current to voltage, i.e.,

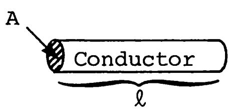
$$G = \frac{1}{R} = \frac{i}{v}\,[\text{mho}(\mho)].$$

2.3.3 RESISTIVITY

ρ, the characteristic of a material which indicates how much a material impedes current flow.

$$\boxed{R = \frac{\rho \ell}{A}}$$ (at constant temperature)

R = resistance in ohms
ℓ = length [m]
A = cross-sectional area [m^2]
ρ = resistivity [Ω-m]



Note: Resistivity is low in a good conductor but high in a poor conductor (insulator).

2.3.4 RESISTOR AND CONDUCTOR COMBINATIONS

For Series Combination of N Resistors

$$R_{eq} = R_1 + R_2 + \ldots + R_N$$

or

$$\frac{1}{G_{eq}} = \frac{1}{G_1} + \frac{1}{G_2} + \ldots + \frac{1}{G_N}$$

For Parallel Combination of N Resistors

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_N}$$

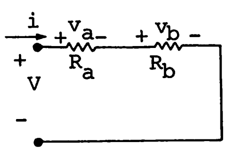
or

$$G_{eq} = G_1 + G_2 + \ldots + G_N$$

2.4 VOLTAGE AND CURRENT DIVISION

2.4.1 VOLTAGE DIVISION

Circuit diagram:

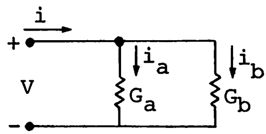


Formula:

$$\boxed{v_b = \frac{R_b}{R_a+R_b}v}$$

2.4.2 CURRENT DIVISION

Circuit diagram:

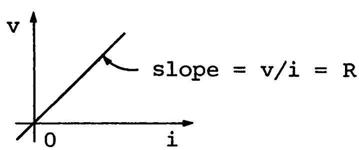


Formula:

$$\boxed{i_b = \frac{G_b}{G_a+G_b}\,i = \frac{R_a}{R_a+R_b}\,i}$$

2.5 OHM'S LAW

The voltage across a conducting material is directly proportional to the current through the material, i.e., $v = Ri$, where R(resistance) is the proportionality constant.



Hence, absorbed power in a resistor is given by

$$\boxed{p = Vi = i^2R = V^2/R.}$$

Note: This power is in the form of heat because a resistor is a passive element; it neither delivers power nor stores energy.

Short Circuit: Circuit as a zero ohms resistance, i.e., voltage across a short circuit = 0.

Open Circuit: Circuit as an infinite resistance, i.e., current across an open circuit = 0.

2.6 KIRCHHOFF'S LAW

2.6.1 KIRCHHOFF'S CURRENT LAW

The algebraic sum of all currents entering a node equals the algebraic sum of all currents leaving it, i.e., for a given node, Σ currents entering = Σ currents leaving or

$$\sum_{n=1}^{N} i_n = 0.$$

2.6.2 KIRCHHOFF'S VOLTAGE LAW

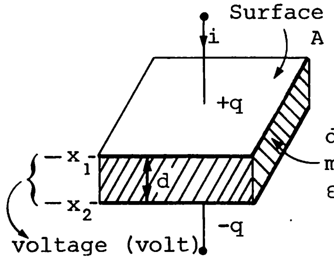
The algebraic sum of all voltages around a closed loop (or path) is zero, i.e., for a closed loop, Σ potential rises = Σ potential drops.

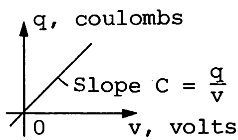
CHAPTER 3

TRANSIENT CIRCUITS

3.1 CAPACITOR AND CORRESPONDING VOLTAGE, CURRENT AND POWER RELATIONSHIPS

3.1.1 PARALLEL PLATE CAPACITOR





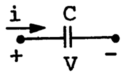
(potential existing between points x_1 and x_2 due to charge accumulation)

Capacitance (C) $= K\varepsilon_0 \frac{A}{d}$ (where $K = \varepsilon/\varepsilon_0 =$ relative dielectric constant) $= \frac{q}{v}$ [Coulombs/volt or farad (F)]

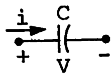
or $C = q(t)/v(t)$ (time variant)

(Note: ε_0 (for air or vaccum) $= 8.854$ pF/m $= \left(\frac{1}{36\pi}\right)$ nF/m)

Circuit Symbol:



or



Capacitor Voltage, Current, Power and Energy:

$$\boxed{v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau}$$

$$\boxed{i(t) = \frac{dq(t)}{dt} = C\,\frac{dv(t)}{dt}}$$

$$p = Cv\left(\frac{dv}{dt}\right)$$

and

$$\boxed{W = \tfrac{1}{2}C\,v^2} \quad = \text{ stored energy [joules]}$$

or

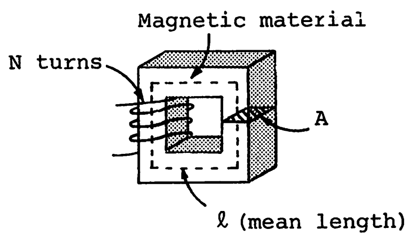
$$W = \frac{Q^2}{2C}$$

Characteristic:

A capacitor acts as an open circuit to dc.

3.2 INDUCTOR AND TRANSFORMER

3.2.1 IRON-CORE INDUCTOR



Inductance (L) = $\frac{\mu N^2 A}{\ell}$ where N = no. of turns of coil
μ = permeability of core
A = cross-sectional area of core
ℓ = mean length of core.

(Note: permeability of vacuum $\mu_0 = 4\pi \times 10^{-7}$.)

Magnetic flux, $\phi(t) = \left(\frac{\mu N^2 A}{\ell}\right) i(t) = L\, i(t)$.

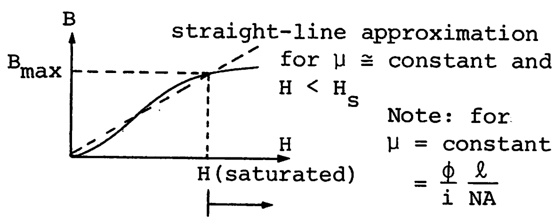
Magnetic field intensity, $H = \frac{Ni}{\ell}$ [A·turn/m].

ℓ = length of material through which ϕ travels.

i = current flowing in the coil.

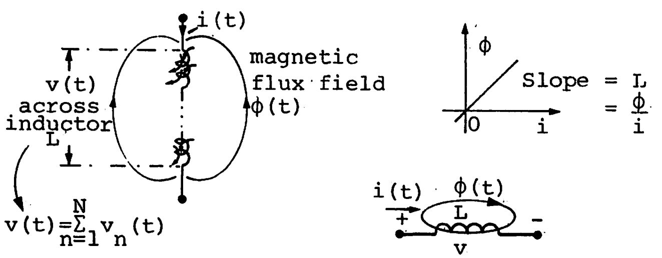
Flux density, $B = \mu H = \phi/A$ [telsa (T) or wb/m^2].

B-H Curve:



3.2.2 INDUCTOR

Concept of Self-Inductance:



circuit symbol

Inductance (L) $= \frac{N d\phi(i)}{di}$ [henry(H) or volt-second/ampere].

Voltage $v(t) = N$(no. of turns of coil) $\times \frac{d\phi(t)}{dt}$
(rate of change of ϕ with respect to time), or

$$\boxed{v(t) = \frac{N d\phi(i)}{di} \quad \frac{di}{dt} = L\frac{di(t)}{dt}}$$

$$\boxed{i(t) = \frac{1}{L}\int_{-\infty}^{t} v\,dt}$$

$$\boxed{W = \frac{1}{2}Li^2}$$

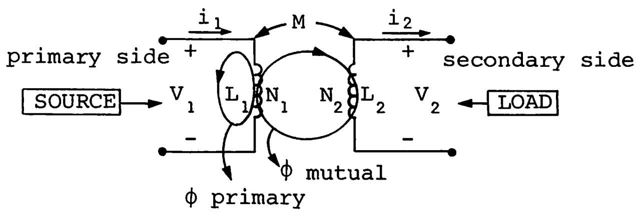
$$p = vi = Li\frac{di}{dt} \text{ [W]}.$$

An inductor acts as a short circuit to dc.

3.2.3 MUTUAL INDUCTANCE

Definition:

The coupling of two coils such that the change of flux produced by one will link the other, resulting in an induced voltage across each coil. (See Fig.)



From the figure above,

$$v_2(t) = \frac{M di_1(t)}{dt} \quad \text{and} \quad v_1(t) = \frac{M di_2(t)}{dt}$$

or, by Faraday's law,

$$v_2(t) = \frac{N_2 d\phi_m}{dt} \quad \text{and} \quad v_1(t) = \frac{N_1 d\phi_p}{dt},$$

where

M = proportionality constant between 2 coils

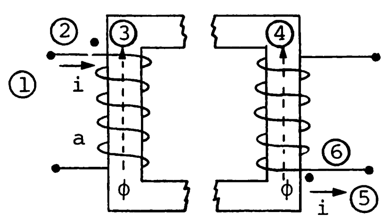
= mutual inductance

= $K\sqrt{L_1 L_2}$ [Henry (H) or volt• second/ampere];

K = coupling coefficient = $\dfrac{\phi_m}{\phi_p}$

3.2.4 DOT NOTATION

Assigning the dots on a pair of coupled coils:



Procedure:

1. Select current direction in one of the coils.

2. Assign a dot to where the current enters the winding. (Note: This is the positive terminal with respect to point a.)

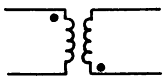
3. Use the right-hand rule to assign flux direction.

4. By Lenz's law, assign opposite flux direction for the second coil.

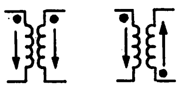
5. Use right-hand rule to assign current direction.

6. Assign a dot to where the current leaves the winding.

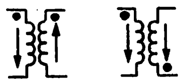
7. Obtain simplified diagram as shown:



Sign of mutual inductance M:



M(+) M(+)

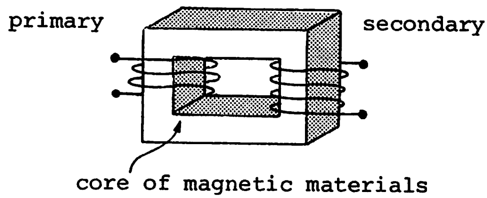


M(-) M(-)

M = + (Both currents pass through coils and are either leaving or entering dots.)

M = - (If arrow indicating current direction through coil is entering the dot for one coil and leaving the dot for another.)

3.2.5 IRON-CORE TRANSFORMER

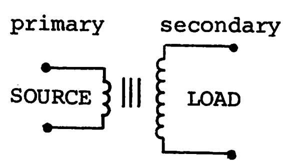


Turns ratio: The determination of how much a transformer steps up or steps down a voltage.

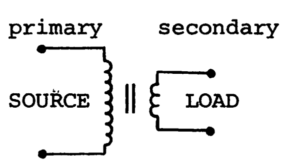
$$\text{Turns ratio} = \frac{\text{no. of turns on the primary } (N_p)}{\text{no. of turns on the secondary} (N_s)}$$

$$\text{or } \frac{V_p}{V_s} = \frac{N_p}{N_s} \text{ and } \frac{I_p}{I_s} = \frac{N_s}{N_p}$$

Voltage step-up transformer $N_p < N_s$:

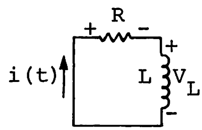


Voltage step-down transformer $N_p > N_s$:



3.3 SIMPLE RL AND RC CIRCUITS

3.3.1 SOURCE FREE RL CIRCUIT



Properties: Assume initially $i(0) = I_0$.

a) $v_R + v_L = Ri + L\frac{di}{dt} = 0.$

b) $i(t) = I_0 e^{-Rt/L} = I_0 e^{-t/\tau}$, τ = time constant $= \frac{L}{R}$

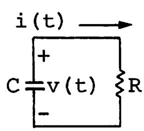
c) Power dissipated in the resistor =

$P_R = i^2 R = I_0^2 R e^{-2Rt/L}.$

d) Total energy in terms of heat in the resistor =

$W_R = \frac{1}{2} L I_0^2.$

3.3.2 SOURCE FREE RC CIRCUIT



Properties: Assume initially $v(0) = V_0$

a) $C\frac{dv}{dt} + \frac{v}{R} = 0.$

b) $v(t) = v(0)e^{-t/RC} = V_0 e^{-t/RC}.$

(Note: $\boxed{RC = \text{time constant} = \tau.}$

c) $\frac{1}{C} \int_{-\infty}^{t} i(\tau)d\tau + i(t)R = 0$

$$i(t) = i(0)e^{\frac{-t}{RC}}$$

3.4 NATURAL AND FORCED RESPONSE OF RL AND RC CIRCUITS

3.4.1 NATURAL RESPONSE

The complementary solution of a linear differential equation.

3.4.2 FORCED RESPONSE

The particular solution of a linear differential equation.

3.4.3 A GENERAL DIFFERENTIAL EQUATION

For: $\frac{di}{dt} + Pi = Q$, where Q = forcing function,

P = general function of time.

Solution: $i = e^{-pt} \int Qe^{pt}dt + Ae^{-pt}.$

Note: For a source free circuit, Q = 0. $\underbrace{i_n = Ae^{-pt}}_{\text{natural response;}}$

and for $Q(t) = \text{const.}$ $\underbrace{i_f = Q/P}_{\text{forced response;}}$

and for a complete response:

$$\boxed{i(t) = \frac{Q}{P} + Ae^{-pt}}$$

Complete response = natural response + forced response

Total solution = complementary solution + particular solution.

3.4.4 PROCEDURE TO FIND THE COMPLETE RESPONSE f(t) OF RL AND RC CIRCUITS WITH DC SOURCES

	RL	RC
1. Simplify the circuit by "killing" all independent sources and determine:	R_{eq}, L_{eq} (* $\tau = L_{eq}/R_{eq}$)	R_{eq}, C_{eq} (* $\tau = R_{eq}C_{eq}$)
2. Consider: → and use dc-analysis to find:	$L_{eq} \sim$ short circuit $i_L(0^-)$	$C_{eq} \sim$ open circuit $v_c(0^-)$
3. Repeat procedure 2 to find the forced response:	i.e., f(t) as $t \to \infty$ $f(\infty)$	
4. Obtain the total response as the sum of the natural and forced responses:	i.e., $f(t) = Ae^{-t/\tau} + f(\infty)$	

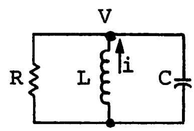
	RL	RC
5. Determine $f(0^+)$ by considering the conditions:	$i_L(0^+) = i_L(0^-)$	$v_C(0^+) = v_C(0^-)$
6. Then:	$f(0^+) = A + f(\infty)$ and $f(t) = [f(0^+) - f(\infty)]e^{-t/\tau} + f(\infty)$	

Killing: setting them equal to zero

3.5 THE RLC CIRCUITS

3.5.1 PARALLEL RLC CIRCUIT (source free)

Circuit diagram:



KCL equation for parallel RLC circuit:

$$\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v\,dt - i(t_0) + C\frac{dv}{dt} = 0;$$

and the corresponding linear, second-order homogeneous differential equation is

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = 0.$$

General solution:

$$V = A_1 e^{S_1 t} + A_2 e^{S_2 t},$$

where

$$S_{1,2} = \frac{-1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2};$$

where α = exponential damping coefficient neper frequency

$$= \frac{1}{2RC}$$

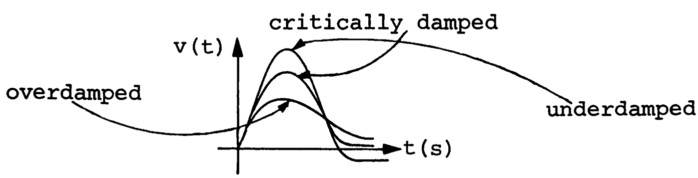
and ω_0 = resonant frequency $= \dfrac{1}{\sqrt{LC}}$

3.5.2 SPECIAL RESPONSE OF PARALLEL RLC CIRCUIT

a) Overdamped	b) Critical damping	c) Underdamped
Condition 1) $\alpha > \omega_0$ or if $LC > 4R^2C^2$	$\alpha = \omega_0$ or $LC = 4R^2C^2$ or $L = 4R^2C$	$\alpha < \omega_0$
2) S_1 and S_2 = negative real numbers, i.e., $\sqrt{\alpha^2 - \omega_0^2} < \alpha$ or $(-\alpha - \sqrt{\alpha^2 - \omega_0^2})$ $< (-\alpha + \sqrt{\alpha^2 - \omega_0^2})$ < 0	$S_1 = S_2 = \alpha$	S_1 and S_2 compose of real and complex quantities.
3) $v(t) \to A_1 e^{S_1 t} \to 0$, as $t \to \infty$	$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$	$v(t) = e^{-\alpha t}(A_1 e^{j\omega_d t}$ $+ A_2 e^{j\omega_d t})$

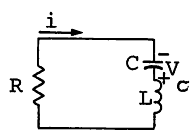
a) Overdamped	b) Critical damping	c) Underdamped
		where $\omega\alpha$ = $\sqrt{\omega_0^2 - \alpha^2}$ = Natural Resonant Frequency or $v(t) = e^{-\alpha t}$ $\Big\{(A_1 + A_2)$ $\left[\dfrac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right]$ + $j(A_1 - A_2)$ $\left[\dfrac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right]\Big\}$ or $v(t) = e^{-\alpha t}$ $\big[(A_1 + A_2)\cos \omega_d t$ + $j(A_1 - A_2)\sin \omega_d t\big]$

Graphic representation:



3.5.3 SERIES RLC CIRCUIT (source free)

Circuit diagram:



KVL equation for series RLC circuit:

$$Ri + \frac{1}{C}\int_{t_0}^{t} i\,dt + L\frac{di}{dt} - v_c(t_0) = 0$$

and the corresponding second-order differential equation in terms of i:

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{c} = 0$$

or in terms of v:

$$LC\frac{d^2 v_c}{dt^2} + RC\frac{dV_c}{dt} - v_c = 0.$$

Special response of series RLC circuit:

a) Overdamped

$$S_1, S_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

or

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where $\alpha = R/2L$,

$\omega_0 = 1/\sqrt{LC}$,

$$i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

b) Critical damping

$$S_1 = S_2 = \alpha$$

$$i(t) = e^{-\alpha t}(A_1 + A_2)$$

c) Underdamped

$$S_{1,2} = -\alpha \pm j\omega_d$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$i(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

3.5.4 COMPLETE RESPONSE OF RLC CIRCUIT

The general equation of a complete response of a second-order system in terms of voltage for an RLC circuit is given by,

$$\boxed{v(t) = \underbrace{V_f}_{} + \underbrace{Ae^{S_1 t} + Be^{S_2 t}}_{}}$$

forced response natural response

(i.e., constant for
DC excitation)

Note: A and B can be obtained by

1) substituting v at $t = 0^+$

2) taking the derivative of the response, i.e.,

$$\frac{dv}{dt} = 0 + S_1 A\, e^{S_1 t} + S_2 B\, e^{S_2 t}, \quad \text{where}$$

$\frac{dv}{dt}$ at $t = 0^+$ is known.

CHAPTER 4

NETWORK THEOREMS

4.1 LINEARITY AND SUPERPOSITION

4.1.1 LINEARITY

1) A linear element: A passive element that can be represented by a linear voltage-current relationship.

2) A linear dependent source: A dependent current or voltage source whose output current or voltage is proportional only to the first power of some current or voltage variable in the circuit or to the sum of such quantities.

3) A linear circuit: A circuit composed entirely of independent sources, linear dependent sources and linear elements.

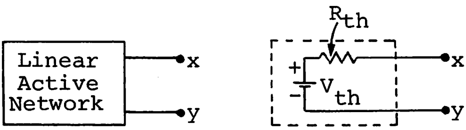
4.1.2 SUPERPOSITION THEOREM

The network response in any linear resistive network with zero initial conditions can be obtained by summing all the individual voltages or currents caused by each independent source acting alone. With all other independent sources set equal to zero, i.e., independent voltage sources are replaced by short circuits and independent current sources are replaced by open circuits.

4.2 THEVENIN'S AND NORTON'S THEOREMS

4.2.1 THEVENIN'S THEOREM

In any linear network, it is possible to replace everything except the load resistor by an equivalent circuit containing only a single voltage source in series with a resistor (R_{th} Thevenin resistance), where the response measured at the load resistor will not be affected.



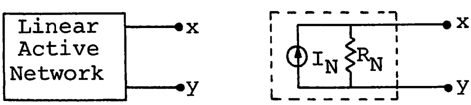
Procedures to find Thevenin equivalent:

1) Solve for the open circuit voltage v_{oc} across the output terminals. $V_{oc} = V_{th}$

2) Place this voltage v_{oc} in series with the Thevenin resistance which is the resistance across the terminals found by setting all independent voltage and current sources to zero. (i.e., short circuits and open circuits, respectively.)

4.2.2 NORTON'S THEOREM

Given any linear circuit, the passive and active components can be converted into an equivalent two-terminal network consisting of a single current source in parallel with a resistor (R_N - Norton resistance).



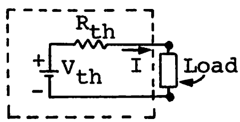
Procedures to find Norton equivalent:

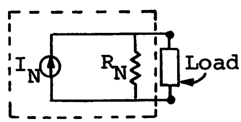
1) Setting all sources to zero (i.e., voltage sources → short circuits, and current sources → open circuits). Then find resulting resistance R_N between the output terminals.

2) I_N is the current through a short circuit applied to the two terminals of the given network.

The Norton's equivalent is obtained by connecting current source I_N and R_N in parallel.

4.3 MAXIMUM POWER TRANSFER THEOREM





Maximum energy transfer occurs between the driving source and the load, if the following condition is satisfied:

$$R_L = R_{th} \quad \text{(for Thevenin equivalent circuit),}$$

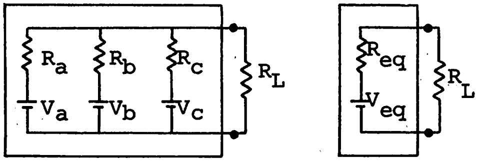
or

$$R_L = R_N \quad \text{(for Norton equivalent circuit).}$$

4.4 MILLMAN'S THEOREM

The theorem states that voltage sources connected in parallel can be reduced to one equivalent circuit.

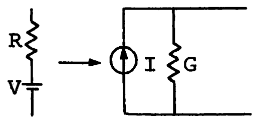
General Procedures: The application of Millman's theorem)



Millman's theorem

1) Given a similar circuit as above, convert all voltage sources to current sources and resistance to conductance into an equivalent parallel circuit as shown below:

2) Combine all parallel current sources and all conductance in parallel.



3) Convert the resulting equivalent current source to a voltage source and the equivalent conductance to an equivalent resistance.

(Note: $R = \frac{1}{G}$ and $V = \frac{I}{G}$.)

4.5 SUBSTITUTION (COMPENSATION) THEOREM

Any network branch voltage or branch current can be replaced by a current or a voltage source which will maintain the same voltage across the chosen branch, and the same current through the branch.

4.6 RECIPROCITY THEOREM

In a linear, single-source network, the current I produced in any branch of the network because of the single voltage source is interchangeable with the location of the voltage source without a change in current.

CHAPTER 5

USEFUL TECHNIQUES OF CIRCUIT ANALYSIS

5.1 MATRICES

5.1.1 A GENERAL FORM OF A RECTANGULAR MATRIX R

$$R = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1N} \\ a_{21} & a_{22} \ldots & a_{2N} \\ a_{M1} & a_{M2} \ldots & a_{MN} \end{bmatrix}$$

a_{ij} = elements of a matrix

i => row

j => column

(Special case: M = N, is a square matrix.)

5.1.2 ADDITION, SUBTRACTION AND MULTIPLICATION OF MATRICES

Consider matrices $X = [x_{ij}]$ and $Y = [y_{ij}]$.

$m \times \ell$ matrix $\qquad \ell \times N$ matrix

Addition and Subtraction:

$$X \pm Y = [x_{ij} \pm y_{ij}]$$

Multiplication:

$$XY = \sum_{k=1}^{\ell} x_{ik}\, y_{kj},$$

where $i = 1,\ldots,M$
and $j = 1,\ldots,N$

Note: $(X+Y)+Z = X+(Y+Z)$

$(X+Y)Z = XZ + YZ$

$(XY)Z = X(YZ)$

$XY \neq YZ$ (in general)

5.1.3 PROPERTIES OF MATRICES

1) Diagonal matrix: $D = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{NN} \end{bmatrix}$

2) Unit or identity matrix:

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

I is also a square matrix.

3) Transpose of a matrix:

$(X^T)^T = X$

$$(X+Y)^T = X^T + Y^T$$

$$(kX)^T = kX^T \quad \text{(where K is a scalar multiple)}$$

$$(XY)^T = Y^T X^T$$

4) Power of matrix:

$$X^n = \underbrace{X\ X\ \dots\ X}_{n \text{ times}}$$

5) Inverse of a matrix:

If X is a square matrix, its inverse is X^{-1} such that

$$X \quad X^{-1} = I.$$

Note: $X^{-1}X = I = XX^{-1}$

$$(XY)^{-1} = Y^{-1}X^{-1}$$

5.2 DETERMINANTS

5.2.1 THE DETERMINANT

In general, given a matrix X i.e.,

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1N} \\ x_{21} & x_{22} & \dots & x_{2N} \\ x_{N1} & x_{N2} & \dots & x_{NN} \end{bmatrix},$$

the determinant of X (Δ_x) can be expressed in terms of minor (Δ_{jk}) as follows:

$$\Delta_x = x_{j1}(-1)^{j+1}\Delta_{j1} + x_{j2}(-1)^{j+2}\Delta_{j2} + \ldots + x_{jN}(-1)^{j+N}\Delta_{jN}$$

$$= \sum_{n=1}^{N} x_{jn}(-1)^{j+n}\Delta_{jn} \text{ — along row j.}$$

$$\Delta_x = x_{1K}(-1)^{1+K}\Delta_{1K} + x_{2K}(-1)^{2+K}\Delta_{2K} + \ldots + x_{NK}(-1)^{N+K}\Delta_{NK}$$

$$= \sum_{n=1}^{N} x_{nk}(-1)^{n+k}\Delta_{nk} \text{ — along column k.}$$

Also, in terms of the cofactor C_{jk} (i.e., $C_{jk} = (-1)^{j+k} \times \Delta_{jk}$),

$$\Delta_x = \sum_{n=1}^{N} x_{jn}C_{jn} = \sum_{n=1}^{N} x_{jk}C_{jk}$$

where j and k are positive integers $\leq$ N.

Cramer's Rule:

In any system of n linear equations of n unknowns, for the kth variable v_k,

$$v_k = \frac{\Delta_k}{\Delta_G}.$$

(Note: Δ_G is the determinant of a conductance matrix.)

5.2.2 SECOND AND THIRD-ORDER DETERMINANT CALCULATIONS

1) Second-order determinant:

$$\Delta_x = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

2) Third-order determinant:

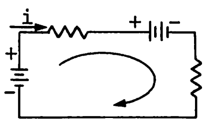
$$\Delta_x = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33}$$

$$= x_{11}[x_{22}x_{33}-x_{23}x_{32}] + x_{12}(x_{23}x_{31}-x_{21}x_{33})+x_{13}(x_{21}x_{32}-x_{22}x_{31})$$

5.3 SINGLE-LOOP AND SINGLE-NODE-PAIR CIRCUIT ANALYSIS

5.3.1 SINGLE-LOOP ANALYSIS

Analytical Procedure:



1) Use an arrow to represent the direction of the unknown current and represent it by i in the circuit.

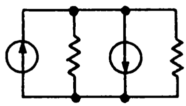
2) For each resistor in the circuit, select a voltage reference.

3) Select a clockwise or counterclockwise direction of movement while applying KVL to the single closed path. Write down positive voltage when a positive terminal is encountered or negative voltage when a negative terminal is encountered.

4) For the resistive elements in the circuit, Ohm's law is applied.

5.3.2 NODE-PAIR ANALYSIS

Analytical Procedure:

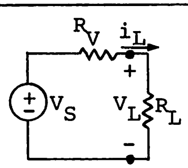


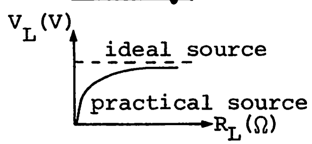
1) Assume a voltage across any element, assigning an arbitrary reference polarity. (Note: Elements connected in parallel have a common voltage across them.)
2) Assign and label the direction of current flow for each resistor.
3) Apply KCL to either of the nodes in the circuit. Note: Apply KCL to the node at which the positive voltage reference has a preferred location.
4) Express the current in each resistor in terms of v and the conductance of the resistor by Ohm's law.

5.4 SOURCE TRANSFORMATIONS

Characteristics of Practical Voltage and Current Sources:

A practical voltage source

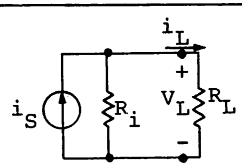


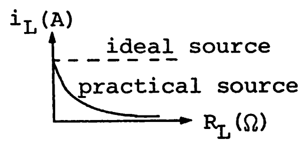


$$V_L = \frac{V_s}{R_v + R_L} R_L$$

$$i_L = \frac{V_s}{R_v + R_L}$$

A practical current source





$$V_L = \frac{R_i R_L}{R_i + R_L} i_s = i_L R_L$$

$$i_L = \frac{R_i}{R_i + R_L} i_s$$

Conditions for equivalence of practical voltage and current source:

1) Each source must produce identical current and identical voltage in any load that is placed across its terminals.

2) $$i_L = \frac{V_s}{R_v + R_L} = \frac{R_i i_s}{R_i + R_L}$$

Hence,

$$R_v = R_i = R_s \quad \text{and} \quad V_s = R_s i_s,$$

where R_s = internal resistance of either practical source.

5.5 NODAL ANALYSIS

5.5.1 NODAL ANALYSIS - FORMAT APPROACH

1) Assign a reference node for the k nodes circuit.

2) For the circuit containing only voltage sources:

   Replace each voltage source by a short circuit without changing the assigned node voltages. Apply KCL at each of the nodes in this modified circuit by using the assigned node to reference voltages. Finally, correspond each source voltage to the variables $v_1, \ldots, v_{k-1}$.

3) For the circuit containing only current sources:

   Apply KCL at each non-reference node. For circuits containing only independent current sources, the conductance matrix can be obtained by using the equation: total current leaving each node (through all conductances) = total source current entering that node. (Note: Put the terms in order from $(v_1, \ldots, v_{n-1})$.

   Finally, for the dependent current source, correspond the source current and the controlling quantity to the variables $v_1, \ldots, v_{k-1}$.

5.5.2 NODAL ANALYSIS - GENERAL APPROACH

1) Convert all voltage sources to current sources.
2) In each network, determine the number of nodes.
3) Choose a reference node and assign voltages to the remaining node.
4) Apply KCL at each node, except at the reference node.
5) Solve the unknown equations for nodal voltages.

5.5.3 MESH ANALYSIS

General Approach:

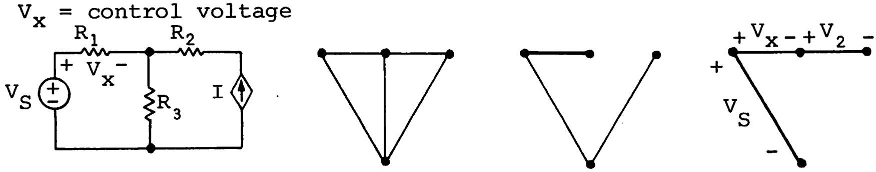
1) Assign closed loops of current called mesh currents, clockwise, to each loop of the circuit.
2) Apply KVL around each closed loop.
3) Solve resulting equations for the assumed loop currents.

Note: Mesh analysis is only applicable to planar network. By definition, a planar network is a circuit diagram on a plane surface such that no branch passes over or under any other branch.

5.6 FREE AND GENERAL NODAL ANALYSIS

A general procedure for writing a set of independent and sufficient nodal equations using the concepts "tree" and "cotree".

V_x = control voltage



1) Draw a graph of the network given and indicate a tree. By definition, a tree is any set of branches which does not contain any loops. It makes connections between nodes, but not necessarily directly.

2) Label and place all voltage sources in the tree.

3) Label and place all current sources in the cotree. By definition, cotrees are those branches which do not belong to the tree.

4) If possible, place and label all control-voltage and control-current in the tree.

5) Complete the tree and assign voltage across each tree branch. (See Fig. for sample illustration)

Concept of link: Any branch belonging to the cotree is a link.

L = number of links = B(number of branches)

- (N(number of nodes)- 1)

= B - N + 1

CHAPTER 6

SINUSOIDAL ANALYSIS

6.1 SINUSOIDAL CURRENT, VOLTAGE AND PHASE ANGLE

6.1.1 SINUSOIDAL FORCING FUNCTION - GENERAL FORM

$$v(t) = V_m \sin(\omega t + \theta)$$

where V_m = maximum value

f = frequency $= \frac{1}{T} = \frac{\text{cycles}}{\text{sec}}$ = hertz (Hz)

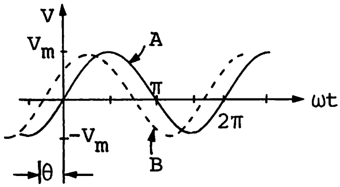
T = period (time duration of 1 cycle = sec)

ω = angular frequency $= 2\pi f = \frac{2\pi}{T} = \frac{\text{radians}}{\text{sec}}$

6.1.2 LEAD AND LAG CONCEPT OF PHASOR ANGLE θ

Let $A = V_m \sin\omega t$ and $B = V_m \sin(\omega t + \theta)$, then

1) B leads A by θ rad.
2) A lags B by θ rad.
3) A leads B by $-\theta$ rad.
4) A leads $V_m \sin(\omega t - \theta)$ by θ rad.



6.1.3 SINUSOIDAL CURRENTS AND VOLTAGES

Voltage across resistance (R), inductance (L) or capacitance (C) if current is indicated as follows:

Element	voltage	$i = I_m \sin \omega t$	$i = I_m \cos \omega t$
R	V_R =	$R I_m \sin \omega t$	$V_R = R I_m \cos \omega t$
L	V_L =	$\omega L I_m \cos \omega t$	$V_L = \omega L I_m(-\sin \omega t)$
C	V_C =	$\frac{I_m}{\omega C}(-\cos \omega t)$	$V_C = \frac{I_m}{\omega C} \sin \omega t$

Current in R, L or C if voltage is indicated as follows:

Element	current	$v = V_m \sin \omega t$	$v = V_m \cos \omega t$
R	i_R =	$\frac{V_m}{R} \sin \omega t$	$i_R = \frac{V_m}{R} \cos \omega t$
L	i_L =	$\frac{V_m}{\omega L}(-\cos \omega t)$	$i_L = \frac{V_m}{\omega L} \sin \omega t$
C	i_C =	$\omega C V_m \cos \omega t$	$i_C = \omega C V_m(-\sin \omega t)$

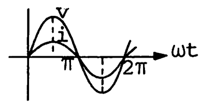
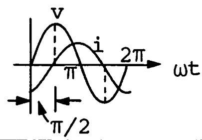
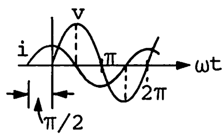
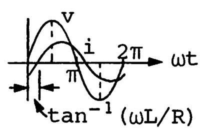
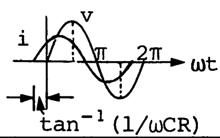
Note:

1) $i_R(t) = \frac{V_R(t)}{R}$, $V_R(t) = i_R(t)R$

2) $i_L(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(\tau)d\tau$, $V_L(t) = L\frac{di_L(t)}{dt}$

3) $i_C(t) = C\frac{dV_c(t)}{dt}$, $V_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau$

6.1.4 CHARACTERISTICS OF PHASE ANGLE IN PURE ELEMENT

Element	Current and voltage phase angle relationship.	Impedance magnitude	Diagram
R	Current and voltage in phase.	R	
L	Current lags the voltage by 90° or π/2 rad.	ωL	
C	Current leads the voltage by 90° or π/2 rad.	$1/\omega C$	
Series RL	Current lags the voltage by $\tan^{-1}(\omega L/R)$.	$\sqrt{R^2+(\omega L)^2}$	
Series RC	Current leads the voltage by $\tan^{-1}(1/\omega CR)$.	$\sqrt{R^2+(1/\omega C)^2}$	

6.2 CONCEPT OF PHASOR

6.2.1 PHASOR NOTATION

In general, the phasor form of a sinusoidal voltage or current is

$$V = V_m \angle\theta \quad \text{and} \quad I = I_m \angle\theta.$$

Thus, for the voltage source $v(t) = V_m \cos\omega t$, the corresponding phasor form is $V_m = \angle\theta^0$. For the current response $i(t) = I_m \cos(\omega t+\theta)$, the corresponding phasor form is $I_m = \angle\theta^0$.

6.2.2 TIME DOMAIN TO FREQUENCY DOMAIN TRANSFORMATION, AND VICE VERSA

Time domain → frequency domain:

i.e., $v(t) = V_m \cos(\omega t + \theta) \rightarrow V = V_m \angle\theta$

1) Assume a sinusoidal function i(t) in the time domain is given. Express i(t) as a cosine wave with a phase angle.

2) Using Euler's identity-- $e^{j\theta} = \cos\theta + j\sin\theta$ --express the cosine wave as the real part of a complex quantity.

3) Drop the Re and the term $e^{j\omega t}$ to obtain the final phasor form (the frequency domain form).

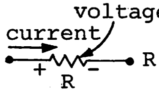
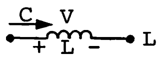
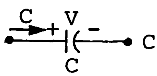
Frequency domain → time domain:

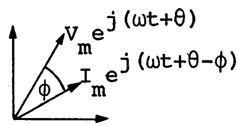
1) Given a phasor current or voltage in polar form in the frequency domain, express the complex expression in exponential form.

2) Multiply the factor $e^{j\omega t}$ to the obtained exponential form.

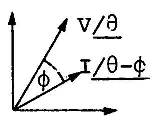
3) Apply the Euler's identity and take the real part of the complex expression to obtain the time-domain representation.

6.2.3 TIME-DOMAIN AND FREQUENCY-DOMAIN RELATIONSHIPS OF VOLTAGE AND CURRENT FOR ELEMENT R, L AND C

Element	Voltage & Current Relationship	
	Time domain	Frequency domain
 R	$v = Ri$	$V = RI$
 L	$v = L\,di/dt$	$V = (j\omega L)I$
 C	$v = \frac{1}{C}\int i\,dt$	$V = \left(\frac{1}{j\omega C}\right) I$



time domain plot



frequency domain plot

(Note: V_m and I_m are $1/\sqrt{2}$ times V and I.)

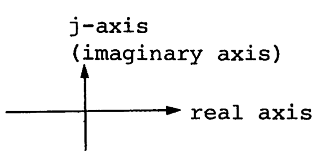
6.3 COMPLEX NUMBERS

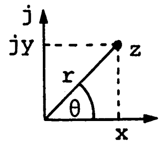
6.3.1 IMAGINARY NUMBERS

$$j = \sqrt{-1}$$

Commonly used forms: $j^2 = -1$, $j^3 = j^2 j = -j$, $j^4 = (j^2)^2 = 1$, $j^5 = j$, etc.

6.3.2 COMPLEX NUMBERS



Rectangular form:	Polar form:	Exponential form:
$Z = x + jy$ $\uparrow$ real, $\uparrow$ imaginary $Z^* = x-jy$ (complex conjugate of Z)	$Z = r(\cos\theta + j\sin\theta)$ where $x = r\cos\theta$ $y = r\sin\theta$ $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1} y/x$	$Z = re^{j\theta}$ (By Euler's identity, i.e., $e^{j\theta} = (\cos\theta + j\sin\theta)$) and $Z^* = re^{-j\theta}$
Note: For x=0, pure imaginary, a point on j-axis. y=0, real number, a point on real axis	 $Z = r\angle\theta =$ $r(\cos\theta + j\sin\theta)$ $Z^* = r\angle -\theta =$ $r(\cos\theta - j\sin\theta)$	

6.3.3 COMPLEX NUMBERS - MULTIPLICATION AND DIVISION

	Multiplication $Z_1 Z_2$	Division Z_1/Z_2
Exponential ($Z=re^{j\theta}$)	$r_1 r_2 e^{j(\theta_1+\theta_2)}$	$\frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$
Polar ($z=r\angle\theta$)	$r_1 r_2 \angle\theta_1+\theta_2$	$\frac{r_1}{r_2} \angle\theta_1 - \theta_2$
Rectangular ($Z=x+jy$)	$x_1 x_2 + jx_1y_2 + jy_1x_2 + j^2 y_1 y_2$	$\frac{(x_1x_2+y_1y_2)+j(y_1x_2-y_2x_1)}{x_2^2+y_2^2}$
[Note: $Z_1=r_1e^{j\theta_1}$ $Z_2=r_2e^{j\theta_2}$]		

6.3.4 POWER OF COMPLEX NUMBER

Given a complex number $Z = x+jy$, where $Z = re^{j\theta}$ ($r\cos\theta=x$ and $r\sin\theta=y$), then

$$Z^n = (re^{j\theta})^n = r^n e^{j\theta n} = r^n(\cos n\theta + j\sin n\theta),$$

where

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ and } r = \sqrt{x^2 + y^2}.$$

6.3.5 ROOTS OF A COMPLEX NUMBER

Given a complex number $Z = x+jy$,

then
$$\sqrt[n]{Z} = \sqrt[n]{r}\,[e^{j\,\theta+2k\pi}]^{1/n}$$
$$= \sqrt[n]{r}\,e^{j\left(\frac{\theta+2k\pi}{n}\right)}$$
$$= \sqrt[n]{r}\,\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + j\sin\left(\frac{\theta+2k\pi}{n}\right)\right],$$

where $k = 0,1,2,\ldots,n-1$.

Note: For any complex number,

$$\theta = \theta + 2k\pi, \quad k = 0,\pm1,\ldots.$$

6.3.6 COMMONLY USED FUNCTIONS OF A COMPLEX NUMBER OF THE FORM: Z = x + jy

a) $\sin z = \sin(x+jy) = \sin x\cos jy + \cos x \sin jy$

$$= \sin x\cosh y + j\cos x\sinh y$$

b) $\log_a z = \log_a(x+jy) = \log_a r + j\theta \log_a e$

(Note: $x+jy = re^{j\theta}$)

c) $e^z = e^{x+jy} = e^x e^{jy} = e^x(\cos y + j\sin y)$

6.3.7 EULER'S THEOREM

The Taylor expansion of $e^{j\theta}$ is given by

$$e^{j\theta} = 1 + j\theta + \frac{j^2\theta^2}{2!} + \ldots + \frac{j^n\theta^n}{n!} + \ldots\ldots$$

$$= \underbrace{[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots]}_{\cos\theta} + j\underbrace{[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots]}_{\sin\theta}$$

Other forms:

1) $\cos\theta = \dfrac{e^{j\theta}+e^{j\theta}}{2}$, $\quad \sin\theta = \dfrac{e^{j\theta}-e^{j\theta}}{2j}$

2) $\cos j\theta = \dfrac{e^{j(j\theta)}+e^{-j(j\theta)}}{2} = \cosh\theta$, $\quad \sin j\theta = \dfrac{e^{j(j\theta)}-e^{-j(j\theta)}}{2j}$

$= j\sinh\theta$

6.4 IMPEDANCE AND ADMITTANCE

6.4.1 IMPEDANCE

Definition:

$$\boxed{\text{impedance } (Z) = \frac{\text{phasor voltage}}{\text{phasor current}} \text{ [ohms].}}$$

Note: Impedance is a complex quantity.

6.4.2 TWO GENERAL FORMS OF AN IMPEDANCE

1) $\boxed{\text{Polar form: } Z = |Z| \angle\theta}$

2) $\boxed{\text{Rectangular form: } Z = R \pm jx}$

R = Resistive component

x = Reactive component

Impedance for elements R, L and C in the frequency domain are expressed as follows:

1) $Z_R = \frac{V}{I} = R$

2) $Z_L = \frac{V}{I} = j\omega L$

3) $Z_C = \frac{V}{I} = \frac{1}{j\omega C}$

Note: $+jx \rightarrow$ inductive reactance, $X_L = \omega L$

$-jx \rightarrow$ capacitive reactance, $X_C = 1/\omega C$

6.4.3 ADMITTANCE

Definition:

$$\boxed{\text{Admittance } (Y) = \frac{1}{Z} = \frac{\text{Phasor current}}{\text{Phasor voltage}} \quad \text{[mhos]}}$$

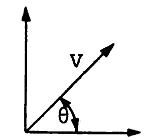
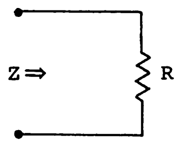
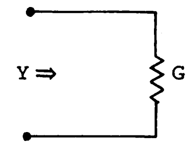
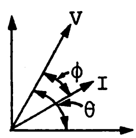
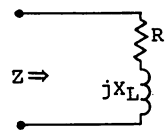
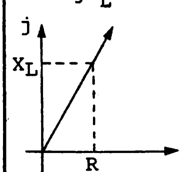
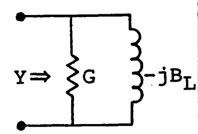
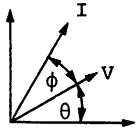
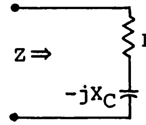
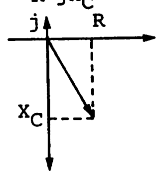
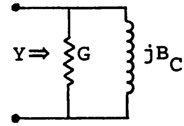
General form of Admittance:

$$\boxed{Y = G \pm jB}$$

Note: positive sign (i.e., $+jB$) $\rightarrow$ capacitive susceptance

negative sign (i.e., $-jB$) $\rightarrow$ inductive susceptance

6.4.4 REPRESENTATION OF IMPEDANCE AND ADMITTANCE IN TERMS OF PHASOR VOLTAGE AND CURRENT

Phaser diagram	Impedance	Admittance
 $V = V\underline{/\theta}$ $i = I\underline{/\theta}$	 $Z = \dfrac{V\underline{/\theta}}{I\underline{/\theta}} = Z\underline{/0°} = R$	 $Y = \dfrac{I\underline{/\theta}}{V\underline{/\theta}} = Y\underline{/0°} = G$
 $V = V\underline{/\theta}$ $i = I\underline{/\theta - \phi}$	 $Z = \dfrac{V\underline{/\theta}}{I\underline{/\theta - \phi}} = Z\underline{/\phi}$ $= R + jX_L$ 	 $Y = \dfrac{I\underline{/\theta - \phi}}{V\underline{/\theta}} = Y\underline{/-\phi}$ $= G - jB_L$
Phaser diagram	Impedance	Admittance
 $V = V\underline{/\theta}$ $i = I\underline{/\theta + \phi}$	 $Z = \dfrac{V\underline{/\theta}}{I\underline{/\theta + \phi}} = Z\underline{/-\phi}$ $= R - jX_C$ 	 $Y = \dfrac{I\underline{/\theta + \phi}}{V\underline{/\theta}} = Z\underline{/-\phi}$ $= G + jB_C$

6.4.5 CONVERSION OF Z TO Y AND VICE VERSA IN POLAR FORM

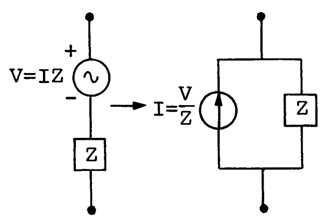
1) $Z = \frac{1}{Y} = R \pm jx = \frac{G}{G^2+B^2} \pm j\frac{-B}{G^2+B^2}$

2) $Y = \frac{1}{Z} = G \pm jB = \frac{R}{R^2+x^2} \pm j\frac{-x}{R^2+x^2}$

6.5 AC ANALYSIS

Procedures similar to DC analysis and theorems are used for AC analysis except that they are in terms of phasor voltages and current (V and I) and impedance (Z).

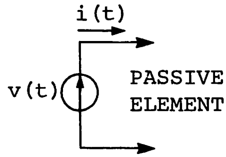
Note: In the case of source conversions, the general format is as follows:



6.6 AVERAGE POWER AND rms VALUES

6.6.1 INSTANTANEOUS POWER

$p = vi$



Note: p = +'ve, energy transfer from source to network.

p = -'ve, energy transfer from network to source.

In a resistive circuit, $p = i^2 R = v^2/R$.

In a inductive circuit, $p = Li\frac{di}{dt} = \frac{1}{L}v\int_{-\infty}^{t} vdt$.

In a capacitive circuit, $p = Cv\frac{dv}{dt} = \frac{1}{C}i\int_{-\infty}^{t} idt$.

6.6.2 AVERAGE POWER

$$\boxed{\text{Average power (P)} = \frac{1}{2} V_m I_m \cos\theta}$$

$$= \underbrace{V_{rms} I_{rms}}_{\text{apparent power}} \cos\theta, \text{ where } V_{rms} = V_m/\sqrt{2}$$

$$I_{rms} = I_m/\sqrt{2}$$

(Note: Rms=effective values)

$$\text{Power factor (pf)} = \frac{\text{average power (P)}}{\text{apparent power } (V_{rms} I_{rms})} = \cos\theta$$

Note: The unit of apparent power is voltamperes (VA). Since $\cos\theta$ has maximum value of 1, the magnitude of the apparent power must be greater than the magnitude of the real power.

6.6.3 SPECIAL CASES OF pf

1) In a sinusoidal case, pf = $\cos\theta$, where θ = angle by which the voltage leads the current = pf angle.

2) For a purely resistive load, voltage and current are in phase, i.e., $\theta = 0$ and pf = 1. Hence, apparent power = average power.

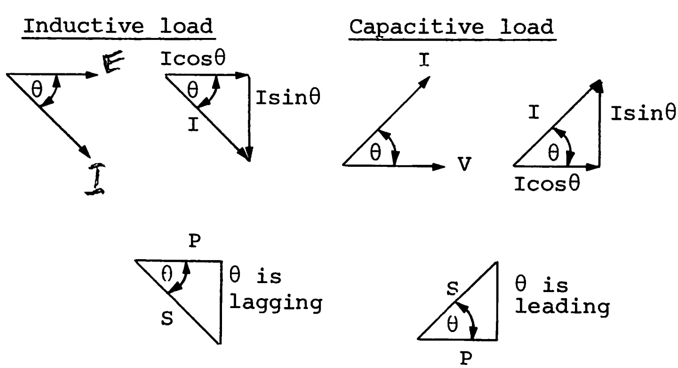
3) For a purely reactive load, the phase difference between the voltage and current is either $+90^0$ or -90^0. Hence, pf = 0.

4) In general networks, 0 < pf < 1.

Summary:

> Reactive power = $V_{rms} I_{rms} \sin\theta$ [voltamperes • reactive (VAR)] = Q
>
> Apparent power = $V_{rms} I_{rms}$ [voltampere(VA)] = S
>
> Average power = $V_{rms} I_{rms} \cos\theta$ = P

6.6.4 POWER TRIANGLE FOR INDUCTIVE AND CAPACITIVE LOAD



6.6.5 COMPLEX POWER

Complex power (P) $= V_{rms} I^*_{rms}$

$$= V_{rms} I_{rms} \, e^{j(\theta_v - \theta_i)}$$

$$= V_{rms} I_{rms} \cos\theta - jV_{rms} I_{rms} \sin\theta$$

$$= P \qquad - \qquad jQ$$

real average power reactive power

Complex Representation of P, Q, S and pf:

$P = \mathrm{Re} V_{rms} I^*_{rms}$

$Q = I_m V_{rms} I^*_{rms}$

$S = \left| V_{rms} I_{rms}{}^* \right|$

$pf = P/S = \cos\theta$

6.6.6 RMS OR EFFECTIVE VALUE

Effective value can be obtained as follows:

1) Square the time function.

2) Take the average value of the squared function over a period.

3) Take the square root of the average of the squared function.

(Note: Effective value = square root of the mean square = rms value, i.e.,

$$I_{eff} = I_{rms} = \sqrt{\frac{1}{T} \int_0^T [i(t)]^2 \, dt}$$

Special cases:

1) Effective value of $a\sin\omega t$ and $a\cos\omega t = a/\sqrt{2}$

2) I_{eff} for sinusoidal current i(t) equals $I_m \cos(\omega t - \theta)$, with

$T = 2\pi/\omega = I_m / \sqrt{2} = 0.707\, I_m$.

The ESSENTIALS of ELECTRIC CIRCUITS II

Staff of Research and Education Association,
Dr. M. Fogiel, Director

This book is a continuation of "*THE ESSENTIALS OF ELECTRIC CIRCUITS I*" and begins with Chapter 7. It covers the usual course outline of Electric Circuits II. Earlier/basic topics are covered in "*THE ESSENTIALS OF ELECTRIC CIRCUITS I*".

Research and Education Association
505 Eighth Avenue
New York, N.Y. 10018

WHAT "THE ESSENTIALS" WILL DO FOR YOU

This book is a review and study guide. It is comprehensive and it is concise.

It helps in preparing for exams, in doing homework, and remains a handy reference source at all times.

It condenses the vast amount of detail characteristic of the subject matter and summarizes the **essentials** of the field.

It will thus save hours of study and preparation time.

The book provides quick access to the important facts, principles, theorems, concepts, and equations of the field.

Materials needed for exams, can be reviewed in summary form – eliminating the need to read and re-read many pages of textbook and class notes. The summaries will even tend to bring detail to mind that had been previously read or noted.

This "ESSENTIALS" book has been carefully prepared by educators and professionals and was subsequently reviewed by another group of editors to assure accuracy and maximum usefulness.

Dr. Max Fogiel
Program Director

CONTENTS

This book is a continuation of *"THE ESSENTIALS OF ELECTRIC CIRCUITS I"* and begins with Chapter 7. It covers the usual course outline of Electric Circuits II. Earlier/basic topics are covered in *"THE ESSENTIALS OF ELECTRIC CIRCUITS I"*.

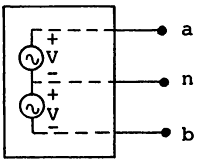
CHAPTER 7

POLYPHASE SYSTEMS

7.1 SINGLE-PHASE, 2-PHASE AND 3-PHASE SYSTEMS

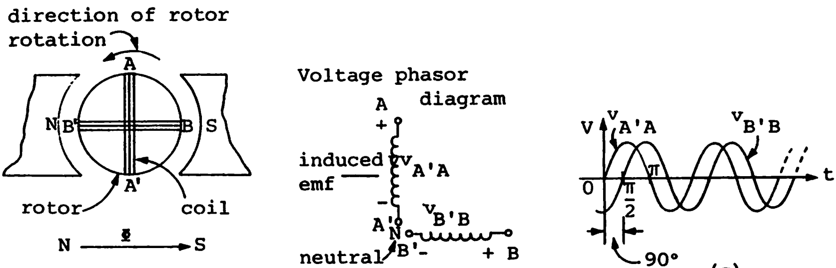
7.1.1 SINGLE-PHASE, THREE-WIRE SYSTEM

The representation of a general single-phase, three-wire system is:



Since $v_{an} = v_{nb} = v$, $v_{ab} = 2v_{an} = 2v_{nb}$.

7.1.2 TWO-PHASE SYSTEM



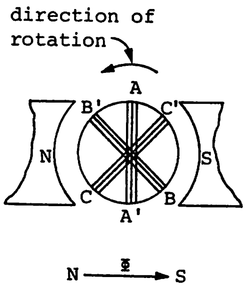
(a) (b) (c)

$$V_{BN} = V_{coil} \angle 0^{\circ}$$

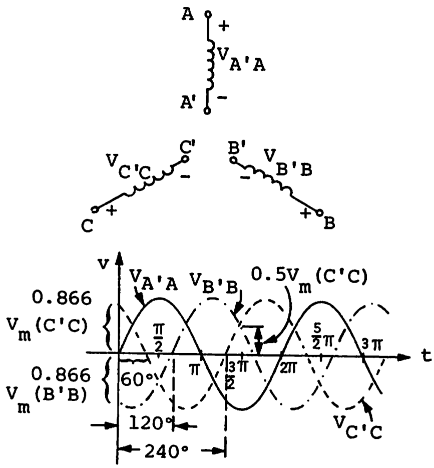
$$V_{AN} = V_{coil} \angle 90^{\circ}$$

$$V_{AB} = V_{AN} + V_{NB} = V_{coil} \angle 90^{\circ} + V_{coil} \angle 180^{\circ} = \sqrt{2}\, V_{coil} \angle 135^{\circ}$$

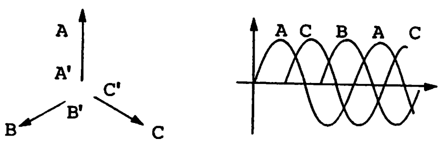
7.1.3 THREE-PHASE SYSTEM



Sequence ABC



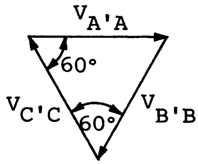
Sequence CBA



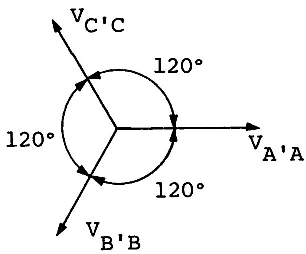
Characteristics

At any instant of time, the summation of all three phase voltages is zero, i.e.,

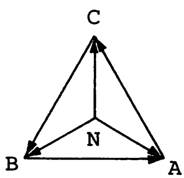
$$\Sigma\,(V_{A'A} + V_{B'B} + V_{C'C}) = 0.$$

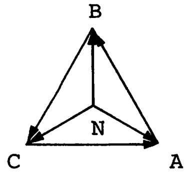


Note: In a three-phase system, the three coils on the rotor are placed 120^{0} apart. (Assume each coil has an equal number of turns.)



7.1.4 THREE-PHASE SYSTEM VOLTAGES

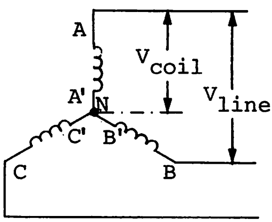
Sequence	(Note: V_L=line voltage)
ABC 	$V_{AN}=(V_L/\sqrt{3})\,\angle 90°$ $V_{BN}=(V_L/\sqrt{3})\,\angle -30°$ $V_{CN}=(V_L/\sqrt{3})\,\angle -150°$ $V_{AB}=V_L\angle 120°$ $V_{BC}=V_L\angle 0°$ $V_{CA}=V_L\angle 240°$

CBA 	$V_{AN}=(V_L/\sqrt{3})\angle -90°$ $V_{BN}=(V_L/\sqrt{3})\angle 30°$ $V_{CN}=(V_L/\sqrt{3})\angle 150°$ $V_{AB}=V_L\angle 240°$ $V_{BC}=V_L\angle 0°$ $V_{CA}=V_L\angle 120°$

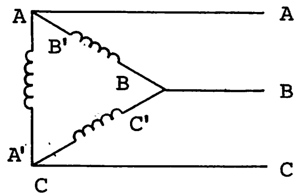
7.2 THE WYE (Y) AND DELTA (Δ) CONNECTIONS

7.2.1 WYE (Y) AND DELTA (Δ) ALTERNATORS

Y-Alternator



Δ-Alternator



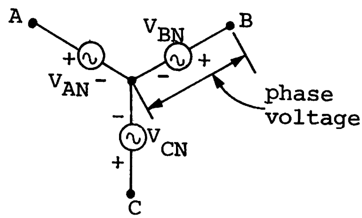
$I_{coil} = I_{line}$ $\quad$ $I_{coil} = \dfrac{1}{\sqrt{3}} I_{line}$

$V_{line} = \sqrt{3}\, V_{coil}$ $\quad$ $V_{line} = V_{coil}$

I_{coil} is more commonly referred to as I_{phase}.

7.2.2 THREE-PHASE Y-Y CONNECTION

Ideal voltage sources connected in Y (three-phase):



7.2.3 CHARACTERISTICS OF BALANCED THREE-PHASE SOURCES

1. $|V_{AN}| = |V_{BN}| = |V_{CN}|$ and $V_{AN} + V_{BN} + V_{CN} = 0$

2. If $V_{AN} = V_P \angle 0^0$ is the reference where V_P = rms is the magnitude of any of the phase voltages, then

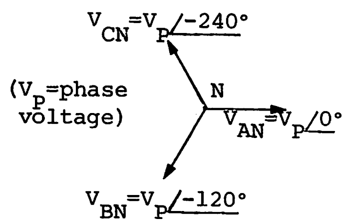
   $V_{BN} = V_P \angle -120^0$ and $V_{CN} = V_P \angle -240^0$ (positive phase or sequence ABC),

   or

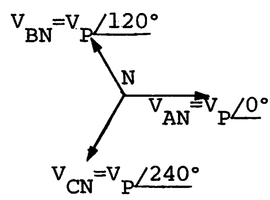
   $V_{BN} = V_P \angle 120^0$ and $V_{CN} = V_P \angle 240^0$ (negative phase or sequence CBA).

Phasor diagrams of:

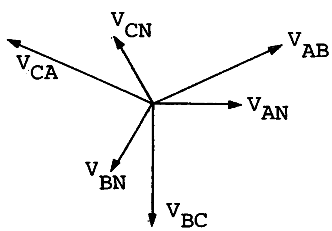
a positive sequence



a negative sequence



3. Phasor diagram of a line and phasor voltage relationship



$$V_{line} = \sqrt{3}\, V_{phase}$$

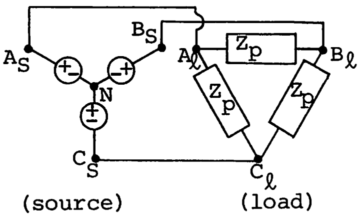
$$V_{AB} = \sqrt{3}\, V_P \;\angle 30^{\circ}$$

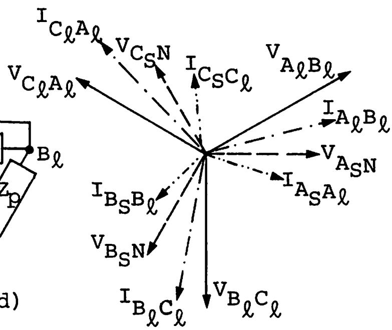
$$V_{BC} = \sqrt{3}\, V_P \;\angle -90^{\circ}$$

$$V_{CA} = \sqrt{3}\, V_P \;\angle -210^{\circ}$$

7.2.4 DELTA CONNECTIONS

A balanced Δ-connected load with Y-connected source:





Phasor Diagram

Given that $V_{phase} = |V_{A_sN}| = |V_{B_sN}| = |V_{C_sN}|$,

assume $V_{line} = |V_{A_sB_s}| = |V_{A_sC_s}| = |V_{C_sA_s}|$

where $V_L = \sqrt{3}\, V_A$ and $V_{A_sB_s} = \sqrt{3}\, V_{A_sN} \;\angle 30^{\circ}$.

Then the phase currents are

$$I_{A_\ell B_\ell} = \frac{V_{A_sB_s}}{Z_p}\,, \quad I_{B_\ell C_\ell} = \frac{V_{B_sC_s}}{Z_p} \text{ and } I_{C_\ell A_\ell} = \frac{V_{C_sA_s}}{Z_p}$$

and the line currents are

$$I_{A_\ell B_\ell} - I_{C_\ell A_\ell} = I_{A_sA_\ell}\,, \text{ etc.}$$

Note: The three-phase currents are equal in magnitude,

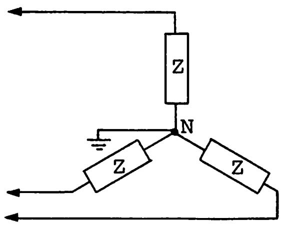
i.e., $I_P = |I_{A_sB_s}| = |I_{B_sC_s}| = |I_{C_sA_s}|$,

$I_L = |I_{A_sA_\ell}| = |I_{B_sB_\ell}| = |I_{C_sC_\ell}|$

and $I_L = \sqrt{3}\, I_P$.

7.3 POWER IN Y- AND Δ-CONNECTED LOADS

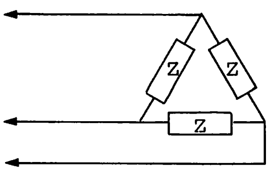
7.3.1 Y-CONNECTED LOAD



For a Y-connected load, the phase power with pf angle $\theta = P_P = V_{phase} I_{line} \cos\theta$ (Note: $V_P I_P = V_P I_L = \frac{V_L I_L}{\sqrt{3}}$

The total power $= P_t = 3P_p$, or $P_t = \sqrt{3}\, V_L I_L \cos\theta$ where $V_L = \sqrt{3}\, V_P$.

Δ-connected load



For a Δ-connected load, the phase power = P_p = $V_L I_P \cos\theta$(Note: $V_P I_P = V_L I_P = V_L \frac{IL}{\sqrt{3}}$).

The total power = $P_t = 3P_p$ or $P_t = \sqrt{3}\, V_L I_L \cos\theta$.

Notice that the total power for any balanced three-phase load is equal to $\sqrt{3}\, V_L I_L \cos\theta$, where S_T (apparent power) = $\sqrt{3}\, V_L I_L$ and θ_T (reactive power) = $\sqrt{3}\, V_L I_L \sin\theta$.

CHAPTER 8

FREQUENCY DOMAIN ANALYSIS

8.1 COMPLEX FREQUENCIES

8.1.1 COMPLEX FREQUENCY

In general, the complex frequency s has the form $s = \delta + j\omega$ which describes an exponentially varying sinusoid. The complex frequency s consists of two parts:

1) The real part, δ, the neper frequency in nepers/sec.

2) The imaginary part, $j\omega$, where ω is the radian frequency in radians/sec.

The real part is related to the exponential variation; the imaginary part is related to the sinusoidal variation.

In general, a function f(t) can be expressed in terms of the complex frequency s as

$$f(t) = k\, e^{st}$$

where k is a complex constant.

The characteristics of the function f(t) relate to the complex frequency s as follows:

s	f(t)
+	increases
-	decreases as t increases
0	constant sinusoidal amplitude

Note: Increasing the magnitude of the real part of s will increase the rate of the exponential increase or decrease. Increasing the magnitude of the imaginary part of s will increase the time function changing rate.

8.2 COMPLEX FREQUENCY IMPEDANCES AND ADMITTANCES

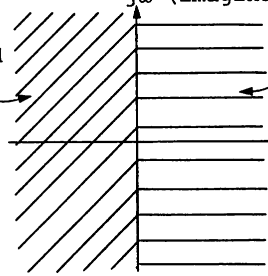
Table of complex frequency impedances and admittances for elements R, L and C:

element	impedance Z(s)	Admittance Y(s)
R	R	$\frac{1}{R}$
L	sL	1/sL
C	1/sC	sC

Note: $Z(s) = \frac{V}{I} = \frac{1}{Y(s)}$

8.3 THE S-PLANE (COMPLEX-FREQUENCY PLANE)

8.3.1 THE S-PLANE



1) A point at the origin. → corresponds to a DC quantity

2) Points on δ -axis → a) $\delta > 0$ ~exponential functions decaying

   b) $\delta < 0$ ~exponential functions increasing

3) Points on jω-axis → purely sinusoidal functions.

4) Points in ▨ (left-hand-region) → describe frequencies of exponentially-decreasing sinusoids.

5) Points in ▤ (right-hand-region) → describe frequencies of exponentially-increasing sinusoids. (i.e., frequencies of positive real parts, time-domain quantities).

8.4 POLES AND ZEROS

Consider a rational function of the form

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}.$$

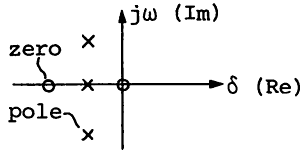
It can be expressed as

$$\boxed{H(s) = k\,\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}}$$

where the zeros of H(s) (i.e., $z_1, z_2, \ldots, z_m$) can be obtained by setting the numerator of H(s) equal to zero.

The poles of H(s) (i.e., $p_1, p_2, \ldots, p_n$) can be obtained by setting the denominator of H(s) equal to zero.

Poles and zeros are indicated in the S-plane (complex-frequency plane) as follows:



Procedures for graphical determination of magnitude and angular variation of frequency-domain function

Step 1: Find all poles and zeros of the frequency-domain function. Indicate them in the S-plane and, for the function to be determined, assign a test point on the S-plane corresponding to the frequency.

Step2: Draw a corresponding arrow from each pole and zero to the test point.

Step 3: Calculate the length and angle of each pole arrow and zero arrow.

Step 4: Determine the magnitude of the frequency-domain function for the assumed frequency of the test point by the following ratio:

$$\frac{\text{product of the zero-arrow lengths}}{\text{product of the pole-arrow lengths}}$$

Step 5: Finally, use the formula

[Sum of zero-arrow angles]-[sum of pole-arrow angles] to obtain the angular variation of the frequency-domain function evaluated at the test point.

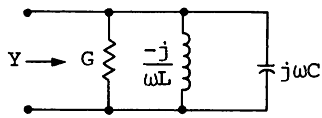
8.5 RESONANCE (SERIES AND PARALLEL)

8.5.1 RESONANCE

In a network, when the voltage and the current at the input terminals are in phase, the network is in resonance.

Note: In resonance, power factor (pf) is unity.

8.5.2 PARALLEL RESONANCE



Characteristics

1) The complex admittance Y is

$$Y = G + j\,(\omega C - 1/\omega L).$$

2) Resonance condition for the circuit:

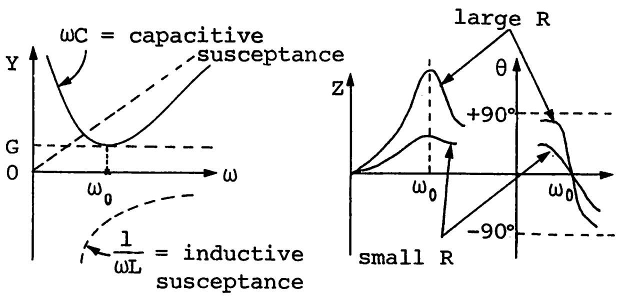
$$\omega C - \frac{1}{\omega L} = 0$$

$$\omega C = \frac{1}{\omega L}$$

$$\omega = \frac{1}{\sqrt{LC}} = \omega_0$$

where ω_0 = resonant frequency

or $f_0 = \dfrac{1}{2\pi\sqrt{LC}} \dfrac{\text{cycles}}{\text{sec}}$



(a) Parallel circuit Y as a function of ω.

(b) Z and θ as a function of ω.

Note: At $\omega < \omega_0$, inductive susceptance > capacitive susceptance and Y is negative.

$\omega > \omega_0$, $\measuredangle$ Z is negative.

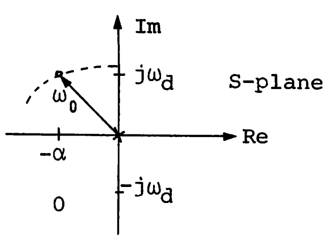
$\omega \to 0$, $\measuredangle$ Z is $+90^0$

4) Pole-zero representation of Y

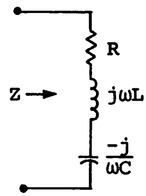
$$Y(s) = K\,\frac{(s+\alpha - j\omega\alpha)(s+\alpha - j\omega\alpha)}{s}$$

where α = exponential damping ratio = $\dfrac{1}{2RC}$,

ω_d = natural resonant frequency = $\sqrt{\omega_0^{\,2} - \alpha^2}$



8.5.3 SERIES RESONANCE



Characteristics

1) The complex impedance Z is

$$Z = R + j(\omega L - \frac{1}{\omega C}).$$

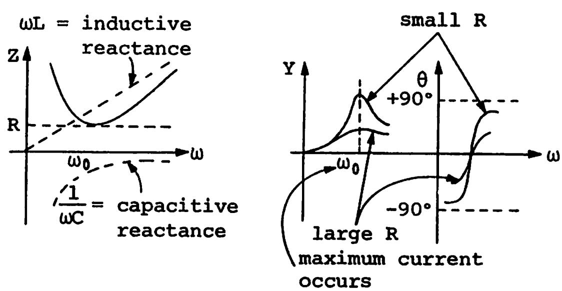
2) Resonance condition for the circuit:

$$\omega L - \frac{1}{\omega C} = 0$$

or

$$\omega = \frac{1}{\sqrt{LC}} = \omega_0$$

Hence, the resonant frequency $(f_0) = \frac{1}{2\pi\sqrt{LC}}$ $\frac{\text{cycles}}{\text{sec}}$



Z, Y, and θ as a function of ω.

When $\omega < \omega_0$, capacitive reactance > inductive reactance and $\measuredangle$ Z is negative.

When $\omega > \omega_0$, inductive reactance > capacitive reactance and $\measuredangle$ Z is positive approaching $+90^0$.

When $\omega \to 0$, $\measuredangle$ Z is -90^0.

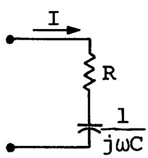
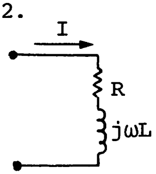
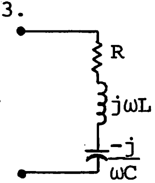
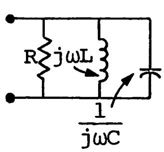
8.6 QUALITY FACTOR - Q

8.6.1 QUALITY FACTOR

By definition, the quality power Q is expressed as

$$Q = 2\pi \times \frac{\text{maximum energy stored}}{\text{total energy lost or dissipated per period}}$$

8.6.2 QUALITY FACTQR Q FOR SOME GENERAL CIRCUITS

CIRCUIT	QUALITY FACTOR, Q
1.  RC series	$Q = 2\pi \times \frac{\frac{1}{2} I^2{}_{max}/\omega^2 C}{(I^2{}_{max}/2)\times R\times T} = \frac{1}{\omega CR}$
2.  RL series	$Q = 2\pi \times \frac{\frac{1}{2} L I^2{}_{max}}{(I^2{}_{max}/2)\times R\times T} = \frac{2\pi f L}{R} = \frac{\omega L}{R}$ $T = \frac{1}{f} = \frac{2\pi}{\omega}$
3.  RLC series	<u>At resonance</u>: $Q_0 = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$ or $Q_0 = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{f_0}{f_2 - f_1} = \frac{f_0}{\text{BW (Bandwidth)}}$ <u>Note</u>: $\frac{1}{2} CV^2{}_{max} = \frac{1}{2} LI^2{}_{max}$ (at maximum)
4.  RLC parallel	<u>At resonance</u>: $Q_0 = \frac{R}{\omega_0 L} = \omega_0 CR$

Note: Both circuits in (3) and (4) store a constant amount of energy at resonance.

8.7 SCALING

The method of scaling is used to ease the numerical calculations during networks analysis. There are basically two types of scaling: magnitude scaling and frequency scaling.

<u>Magnitude scaling</u>

A factor of K_m is increased for the impedance (z) of a two-terminal network with the frequency remaining constant, i.e.,

$R \to K_m R$,

$C \to C/K_m$ and

$L \to K_m L$.

<u>Frequency scaling</u>

A factor of K_f is increased for the frequency at any impedance, i.e.,

$R \to R$,

$C \to C/K_f$ and

$L \to L/K_f$.

CHAPTER 9

STATE - VARIABLE ANALYSIS

9.1 STATE - VARIABLE METHOD

Given a circuit with energy-storing elements (i.e., a capacitor and inductor), the circuit can be analyzed by using the state-variable method. A hybrid set of variables is selected (including capacitor voltages and inductor currents) to describe the energy state of the system.

By using a set of state variables, a set of n first-order, simultaneous differential equations can be obtained from the given nth-order differential equations (i.e., state equations).

9.1.1 CONDITIONS FOR THE STATE-VARIABLE METHOD

1) The state equation must be expressed in normal form, i.e., the derivative of each state variable must be expressed in terms of a linear combination of all the state variables and forcing functions.

2) The equations describing the derivatives must be of the same order as the state variables appearing in each equation.

9.2 STATE EQUATIONS FOR n-th ORDER CIRCUITS

9.2.1 STATE EQUATIONS FOR THE FIRST AND SECOND ORDER CIRCUITS

1) First-order circuit:

State equation

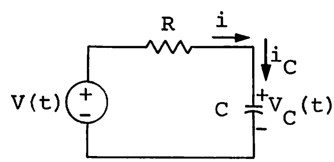
$$\dot{x}(t) = \frac{1}{RC}\,[v(t)-x(t)]$$

1) x(t) represents the state variable $v_c(t)$.

2) $v(t) = Ri(t) + v_c(t),$

$$\uparrow$$

$$i(t) = ic(t) = C\,\frac{dv_c}{dt}$$



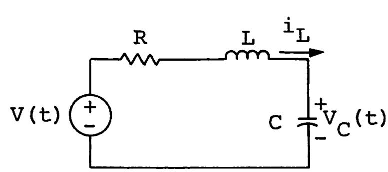
2) Second-order circuit

$$\dot{x}_1(t) = -\frac{R}{L}\,x_1(t) - \frac{1}{L}\,x_2(t) + \frac{1}{L}\,v(t)$$

$$\dot{x}_2(t) = \frac{1}{C}\,x_1(t)$$

$x_1(t) = i_L(t)$

$x_2(t) = v_C(t)$



9.2.2 GENERAL STEPS TO OBTAIN THE STATE EQUATIONS FOR A LINEAR TIME-INVARIANT NETWORK

General steps to obtain the state equations for a linear time-invariant network

Method 1

Step 1: Assign state variables for the voltage across each capacitor and for the current through each inductor.

Step 2: Apply KVL and KCL to obtain a set of linear independent equations for each capacitor and inductor.

Step 3: Rearrange the equations obtained in step 2 so that all other variables in the network are in terms of the chosen state variables.

Step 4: Consider all the equations obtained in step 2 and 3. Simplify them so that the equations are expressed only in terms of the state variables and their corresponding derivatives. Therefore, all network variables not chosen as state variables are eliminated.

Step 5: Rearrange the equations obtained in step 4 in the compact form

$\dot{x}(t) = Ax(t) + bu(t)$ (the normal-form equation),

where $\dot{x}(t)$ = derivative of $x(t) = \frac{dx(t)}{dt}$,

$x(t)$ = n-column vector of all the state variables $(x_1, x_2, \ldots x_n)$ chosen in step 1 (i.e., $x(t) \triangleq [x_1(t) + x_2(t) + \ldots + x_n(t)]^T$,

A = constant $n \times n$ square matrix = system matrix (i.e., $A \triangleq (a_{ij})$),

b = an n-column vector, and bu(t) = the forcing function vector due to the independent sources (i.e., $b \triangleq [b_1, b_2, \ldots, b_n]$).

Therefore,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \vdots \\ A_{n1} & \cdots & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} u(t)$$

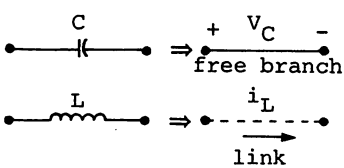
9.3 NORMAL - FORM EQUATIONS

9.3.1 GENERAL PROCEDURES TO OBTAIN A SET OF NORMAL-FORM EQUATIONS

Method 2

Step 1: Obtain a normal tree for the given network using the nodal analysis as outlined in Chapter 5.

Step 2: Assign state variables for the voltage across each capacitor and the current through each inductor correspondingly, i.e.,



or, for the resistive tree branches or links, indicated by using a new voltage or current variable (if necessary).

Step 3: a) For each capacitor, apply KCL (as outlined in Chapter 2) to write a set of equations.

b) For each inductor, repeat part (a) but use KVL instead.

c) If any new voltage and current variables were assigned to the resistors, write the equation for R by using KCL and KVL. Then express v_R and i_R in terms of the state variables and source quantities. Otherwise, skip this step.

Step 4: Put all the equations obtained in step 3 in order and rearrange to obtain the normal-form equations.

9.4 STATE TRANSITION MATRIX- e^{At}

9.4.1 STATE TRANSITION - e^{At}

Let us represent the state equations in the normal form

$$\dot{x}(t) = Ax(t) + bu(t) \quad (\text{At } t=t_0,\ x(t_0) = x_0.) \qquad (1)$$

The solution of the matrix state equation (in (1)) is given by

$$\boxed{x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}bu(\tau)d\tau\ ,}$$

where $e^{At} \triangleq$ the state transition matrix, which describes the change of state of the system from zero to the state at time t where $e^{A(t-t_0)}$ is e^{At} evaluated at $t = t - t_0$.
(Note: $x(t)$ and $e^{A(t-t_0)}bu(\tau)$ are n-column vectors.)

To determine the state transition matrix e^{At}, the Cayley-Hamilton theorem is applied.

9.4.2 CAYLEY-HAMILTON THEOREM

1) By definition,

$$e^{At} \stackrel{\Delta}{=} \mu_0(t)I + \mu_1(t)A + \mu_2(t)A^2 + \ldots + \mu_{n-1}(t)A^{n-1} \quad (2)$$

where A is an n × n square matrix and $\mu_0(t)\ldots\mu_n(t)$ are scalar functions of time.

2) For equation (2) to hold, the following conditions must be satisfied:

a) I = unity matrix.

b) The characteristic equation of the matrix A equals

$$\mathrm{Det}[A - sI] = 0,$$

where Si, i = 1,2,...,n of A are the roots of the characteristic nth-order polynomial equation and are called the eigenvalues of A.

9.4.3 GENERAL PROCEDURES TO OBTAIN THE STATE TRANSITION MATRIX e^{At}, GIVEN MATRIX A

Step 1: Obtain a matrix in the form A - sI.

Step 2: Equate: Det[A - sI] = 0 and solve for the roots (i.e., Si, i = 1,2,...,n) of the characteristic equation.

Step 3: Express each root in n equations of the form $e^{tS_i} = \mu_0 + \mu_1 S_i + \ldots + \mu_{n-1} S_i^{n-1}$, and solve the scalar time functions: $\mu_0, \ldots, \mu_{n-1}$.

Step 4: Obtain the state transition matrix by substituting the time functions obtained in step 3 into Equation (2).

CHAPTER 10

FOURIER ANALYSIS

10.1 TRIGONOMETRIC FOURIER SERIES

10.1.1 DIRICHLET CONDITIONS FOR THE EXISTENCE OF A FOURIER SERIES

If $f(t)$ is a bounded periodic function of period T (i.e., $f(t+T) = f(t)$) and if $f(t)$ satisfies these Dirichlet conditions:

1) $f(t)$, if discontinuous, has a finite number of discontinuities in any period T;

2) $f(t)$ has a finite average value over period T;

3) The number of maxima and minima of $f(t)$ in any period T is finite

then $f(t)$ may be represented by a trigonometric Fourier series as described below:

10.1.2 TRIGONOMETRIC FORM OF A FOURIER SERIES

1) The function $f(t)$ is expressed over any interval $(t_0, t_0+2\pi/\omega_0)$ as

$$f(t) = a_0 + a_1\cos\omega_0 t + a_2\cos 2\omega_0 t + \ldots + b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad (t_0 < t < t_0 + 2\pi/\omega_0)$$

where ω_0 = fundamental frequency $= 2\pi/T$

and $$a_0 = \frac{1}{T} \int_{t_0}^{(t_0+T)} f(t)dt$$ (where t_0 is assumed to be zero generally)

$$a_n = \frac{2}{T} \int_{t_0}^{(t_0+T)} f(t) \cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{(t_0+T)} f(t) \sin n\omega_0 t \, dt$$

10.1.3 SPECIAL INTEGRATION PROPERTIES FOR SINE AND COSINE

1) $$\int_0^T \sin^2 n\omega_0 t \, dt = \frac{T}{2}$$

$$\int_0^T \cos^2 n\omega_0 t \, dt = \frac{T}{2}$$

2) $$\int_0^T \sin n\omega_0 dt = \int_0^T \cos n\omega_0 t \, dt = 0$$

3) $$\int_0^T \sin k\omega_0 t \cos n\omega_0 t \, dt = \int_0^T \sin k\omega_0 t \sin n\omega_0 t \, dt$$

$$= \int_0^T \cos k\omega_0 t \cos n\omega_0 t = 0$$

for $k \neq n$.

10.2 EXPONENTIAL FOURIER SERIES

A given function g(t) can be expressed as a linear combination of exponential functions over the period t_0, $t_0+2\pi/\omega_0$, as follows:

$$g(t) = \ldots + G_{-n}e^{-jn\omega_0 t} + \ldots + G_{-2}e^{-j2\omega_0 t} + G_{-1}e^{-j\omega_0 t}$$

$$+ G_0 + G_1 e^{j\omega_0 t} + G_2 e^{j2\omega_0 t} + \ldots + G_n e^{jn\omega_0 t} + \ldots$$

$$= \sum_{n=-\infty}^{\infty} G_n e^{jn\omega_0 t} \quad (t_0 < t < t_0 + 2\pi/\omega_0),$$

it is assumed that $t_0=0$.

(Note: $T = 2\pi/\omega_0$.)

Therefore,

$$\boxed{G_n = \frac{1}{T}\int_0^T g(t)e^{-jn\omega_0 t}\,dt}$$

and

$$\boxed{G_0 = \frac{1}{T}\int_0^T g(t)\,dt}$$

Relationships between trigonometric and exponential Fourier series

Trigonometric series		Exponential series
a_0	=	G_0
a_n	=	$G_n + G_{-n}$
b_n	=	$j(G_n - G_{-n})$
$\frac{1}{2}(a_n - jb_n)$	=	G_n

10.3 COMPLEX FORM OF A FOURIER SERIES

The complex form of a Fourier series is given as

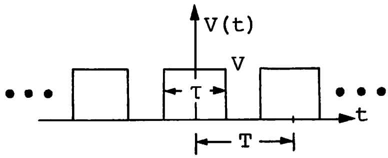
$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where C_0 is a complex constant $= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)e^{-jn\omega_0 t}\, dt$,

for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

Note: $|C_n| = |C_{-n}|$, since $C_{-n} = C_n^*$.

10.3.1 SPECIAL CASE



Given a train of rectangular pulses with period T, i.e.,

$$v(t) = \begin{cases} V & -\tau/2 < t < \tau/2 \\ 0 & \tau/2 < t < T - \tau/2, \end{cases}$$

then

$$C_n = \frac{1}{T} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{2A}{n\omega_0 T} \frac{(e^{jn\omega_0 \tau/2} - e^{-jn\omega_0 \tau/2})}{2j}$$

$$= \frac{A\tau}{T} \left[\frac{\sin(\frac{1}{2}n\omega_0 \tau)}{(\frac{1}{2}n\omega_0 \tau)} \right] = \frac{A\tau}{T} \mathrm{Sa}(\tfrac{1}{2}n\omega_0 \tau),$$

where $\frac{\sin x}{x}$ = sampling function = Sa(x).

$$\text{Thus,}\quad v(t) = \frac{A\tau}{T}\sum_{n=-\infty}^{\infty} Sa(\tfrac{1}{2}n\omega_0\tau)e^{jn\omega_0 t}.$$

10.4 WAVEFORM SYMMETRY PROPERTIES

Waveform symmetry	Properties
1) Even symmetry (i.e., cosine function)	a) $g(t) = g(-t)$ b) $b_n = 0$ c) $a_n = \frac{4}{T}\int_0^{T/2} g(t)\cos n\omega_0 t\,dt$
2) Odd symmetry (i.e., sine function)	a) $g(t) = -g(-t)$ b) $a_0 = a_n = 0$ c) $b_n = \frac{4}{T}\int_0^{T/2} g(t)\sin n\omega_0 t\,dt$
3) Half wave symmetry	a) $g(t) = -g(t+T/2)$ where T = period b) For n = odd, $a_n = \frac{4}{T}\int_0^{T/2} g(t)\cos n\omega_0 t\,dt$

Waveform symmetry	Properties
3) Half wave symmetry	$b_n = \frac{4}{T}\int_0^{T/2} g(t)\sin n\omega_0 t\,dt$ n = even, $a_n = b_n = 0$
4) Half wave and even symmetry	a) For n = odd, $a_n = \frac{8}{T}\int_0^{T/4} g(t)\cos n\omega_0 t\,dt$ $b_n = 0$, n = even, $a_n = b_n = 0$
5) Half wave and odd symmetry	a) For n = odd, $a_n = 0$ and $b_n = \frac{8}{T}\int_0^{T/4} g(t)\sin n\omega_0 t\,dt$ n = even, $b_n = a_n = 0$

10.5 THE FOURIER TRANSFORM

By definition, the Fourier transform of g(t) is

$$\boxed{F\{g(t)\} = G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt}$$

The inverse Fourier transform of F(ω) is

$$\boxed{F^{-1}\{G(\omega)\} = g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega ,}$$

as long as $\int_{-\infty}^{\infty} |f(t)|dt$ converges (i.e., $\int_{-\infty}^{\infty} |f(t)| \, dt < \infty$).

Thus, g(t) and G(ω) are called the Fourier transform pair.

Note: The Fourier transform of g(t) can also be expressed in terms of sine and cosine by using the Euler's identity:

$$e^{-j\omega t} = \cos \omega t - j\sin \omega t.$$

Hence,

$$G(\omega) = \overbrace{\int_{-\infty}^{\infty} g(t)\cos\omega t \, dt}^{R(\omega)} - j \overbrace{\int_{-\infty}^{\infty} g(t)\sin\omega t \, dt}^{I(\omega)}$$

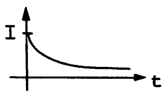
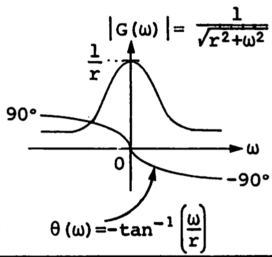
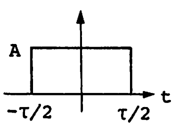
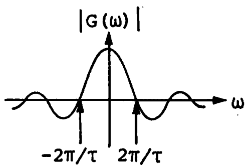
$$= |G(\omega)| \;\angle\theta(\omega)$$

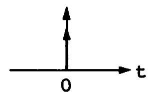
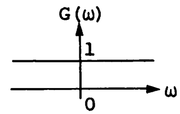
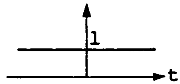
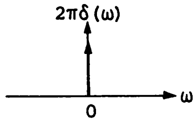
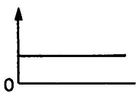
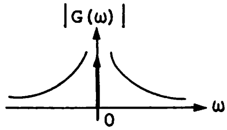
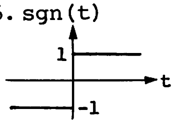
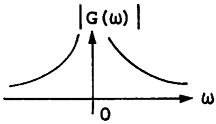
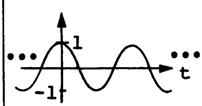
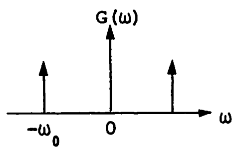
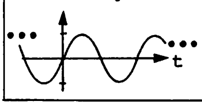
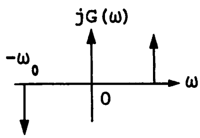
where $|G(\omega)| = [R^2(\omega)+I^2(\omega)]^{\frac{1}{2}}$

and $\theta(\omega) = \tan^{-1}\left(\frac{I(\omega)}{R(\omega)}\right)$

10.5.1 SOME USEFUL FOURIER TRANSFORM PAIRS

$$g(t) \iff G(\omega) = F\{g(t)\}$$

1. $e^{-it}u(t)$ 	$\frac{1}{r+j\omega}$	
2. $A\{u(t+\tau/2)-u(t-\tau/2)\}$ 	$\frac{A\tau\sin(\omega\tau/2)}{(\omega\tau/2)}$ $=A\tau Sa(\omega\tau/2)$	

3. $\delta(t)$ 	1	
4. 1 	$2\pi\delta(\omega)$	
5. $u(t)$ 	$\pi\delta(\omega) + \frac{1}{j\omega}$	
6. $\mathrm{sgn}(t)$ 	$\frac{2}{j\omega}$	
7. $\cos\omega_0 t$ 	$\pi\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\}$	
8. $\sin\omega_0 t$ 	$j\pi\{\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\}$	

10.6 PARSEVAL'S IDENTITY

If $G(\omega) = F\{g(t)\}$, then

$$\boxed{\int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega .}$$

In general, if $G(\omega) = F\{g(t)\}$ and $H(\omega) = F\{h(t)\}$, then

$$\int_{-\infty}^{\infty} g(t)h^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)H^*(\omega)d\omega ,$$

where * denotes the complex conjugate.

Note: $G(-\omega) = G^*(\omega)$

10.7 CONVOLUTION THEOREM FOR THE FOURIER TRANSFORMS

Let $F(\omega) = F\{f(t)\}$ and $G(\omega) = F\{g(t)\}$. Then the convolution of f and g (i.e., $f * g$) is defined as

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{j\omega t}\, d\omega$$

Hence,

$$\boxed{F\{f * g\} = F(\omega)G(\omega) = F\{f\}\, F\{g\}.}$$

CHAPTER 11

LAPLACE TRANSFORMATION

11.1 DEFINITION OF LAPLACE TRANSFORM

By definition,

$$\boxed{L\{g(t)\} = G(s) = \int_0^{\infty} e^{-st}\, g(t)dt}$$

where g(t) is a function of the real variable t, and s is a complex variable defined as $s = \delta + j\omega$. The function g(t) is called the original function and the function G(s) is called the image function.

The transformation of a time domain function into a complex frequency domain function is the operation L{g(t)}.

In order for the Laplace transform to be valid, the following conditions must be satisfied:

1) If the integral in eq.(1) converges for a real $s = s_0$, i.e.,

$$\lim_{\substack{A \to 0 \\ B \to \infty}} \int_A^B e^{-s_0 t}\, g(t)dt \quad \text{exists,}$$

then it converges for all s with $\mathrm{Re}(s) > s_0$, and the image function is a single valued analytic function of s in the half-plane $\mathrm{Re}(s) > s_0$.

2) g(t) is a piecewise continuous function.

1) It should be noted that in specifying the Laplace transform of a signal, both the algebraic expression and the range of values of s for which this expression is valid is required.

2) The range of values s for which the integral defining the Laplace transform converges is referred to as the region of convergence.

11.2 DEFINITION OF THE INVERSE LAPLACE TRANSFORM

If $L\{g(t)\} = G(s)$, then $g(t) = L^{-1}\{G(s)\}$ is the inverse Laplace transform of G(s). L^{-1} is called the inverse Laplace transform operator.

11.3 COMPLEX INVERSION FORMULA

The inverse Laplace transform of G(s) can be found directly by methods of complex variable theory. The result is

$$\boxed{g(t) = \frac{1}{2\pi j}\int_{\delta-j\infty}^{\delta+j\infty} e^{st}G(s)ds = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{\delta-jT}^{\delta+jT} e^{st}G(s)ds}$$

where δ is chosen such that all the singular points of G(s) lie to the left of the line Re(s) = δ in the complex s-plane.

11.4 GENERAL LAPLACE TRANSFORM PAIRS

SIGNAL	TRANSFORM	REGION OF CONVERGENCE
$\delta(t)$	1	All s
$u(t)$	$1/s$	$\text{Re}\{s\} > 0$
$-u(-t)$	$1/s$	$\text{Re}\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$1/s^n$	$\text{Re}\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$1/s^n$	$\text{Re}\{s\} < 0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\text{Re}\{s\} > \alpha$
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\text{Re}\{s\} < -\alpha$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\text{Re}\{s\} > -\alpha$
$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\text{Re}\{s\} < -\alpha$
$\delta(t-T)$	e^{-sT}	All s
$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\text{Re}\{s\} > -\alpha$
$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\text{Re}\{s\} > -\alpha$

11.5 OPERATIONS FOR THE LAPLACE TRANSFORM

1) Linearity of the Laplace Transform:

If $x_1(t) \overset{L}{\longleftrightarrow} X_1(s)$ with region of convergence R_1

and $x_2(t) \overset{L}{\longleftrightarrow} X_2(s)$ with region of convergence R_2

then

$$\boxed{a x_1(t) + b x_2(t) \overset{L}{\longleftrightarrow} a X_1(s) + b X_2(s)}$$

with region of convergence containing $R_1 \cap R_2$

2) Time Shifting:

If $x(t) \overset{L}{\longleftrightarrow} X(s)$ with region of convergence (ROC) $= R$, then,

$$\boxed{x(t-t_0) \overset{L}{\longleftrightarrow} e^{-st_0} X(s) \quad \text{with ROC} = R}$$

3) Shifting in the s-Domain:

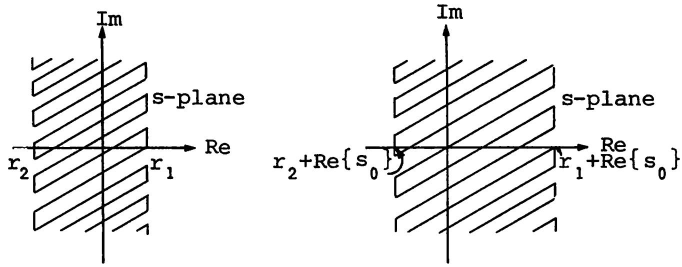
If $x(t) \overset{L}{\longleftrightarrow} X(s)$ ROC $= R$,

then,

$$\boxed{e^{s_0 t} x(t) \overset{L}{\longleftrightarrow} X(s-s_0) \text{ with ROC } R_1 = R + \mathrm{Re}\{s_0\}}$$

Note: The ROC associated with $X(s-s_0)$ is that of $X(s)$, shifted by $\mathrm{Re}\{s_0\}$. Thus, for any value s that is in R, the value $s + \mathrm{Re}\{s_0\}$ will be in R_1.

For example



4) Time Scaling:

If $x(t) \overset{L}{\longleftrightarrow} X(s) \qquad ROC = R,$
then

$$\boxed{x(at) \overset{L}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{with ROC } R_1 = \frac{R}{a}}$$

5) Convolution Property:

If $x\ (t) \overset{L}{\longleftrightarrow} X_1(s) \qquad ROC = R_1$

(and)

$x_2(t) \overset{L}{\longleftrightarrow} X_2(s) \qquad ROC = R_2\ ,$

then

$$\boxed{x_1(t) * x_2(t) \overset{L}{\longleftrightarrow} X_1(s)X_2(s) \quad \text{with ROC containing } R_1 \cap R_2}$$

6) Differentiation in the Time Domain

If $x(t) \overset{L}{\longleftrightarrow} X(s) \qquad ROC = R,$

then

$$\boxed{\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s) \quad \text{with ROC containing } R}$$

7) Differentiation in the s–domain:

Given:
$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

Differentiating both sides:

$$\frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} (-t)x(t)e^{-st}\,dt$$

Hence,

$$\boxed{-t\,x(t) \overset{L}{\longleftrightarrow} \frac{dX(s)}{ds} \qquad \text{ROC} = R.}$$

8) Integration in the Time Domain:

If $x(t) \overset{L}{\longleftrightarrow} X(s)$ ROC $= R$,

then

$$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s} \qquad \text{ROC contains } R \cap \{\mathrm{Re}\{s\} > 0\}}$$

11.6 HEAVISIDE EXPANSION THEOREM

By the theorem,

$L^{-1}\left\{\frac{p(s)}{q(s)}\right\}$, where $q(s)=(s-a_1)(s-a_2)\ldots(s-a_m)$ and $p(s)$ = a polynomial of degree $<$ m.

$$= \sum_{n=1}^{m} \frac{p(a_n)}{q'(a_n)} e^{a_n t}$$ (i.e., Heaviside Expansion Formula)

11.7 FINAL AND INITIAL VALUE THEOREM

Initial value theorem:

$$\boxed{g(0^+) = \lim_{s\to\infty} \{sG(s)\}}$$

Final value theorem:

$$\boxed{g(\infty) = \lim_{s\to 0} sG(s).}$$

Note: All poles of $sG(s)$ lie in the left-hand side of the complex s-plane.

CHAPTER 12

TWO PORT NETWORK PARAMETERS

12.1 Z - PARAMETERS

12.1.1 IMPEDANCE OR Z-PARAMETERS

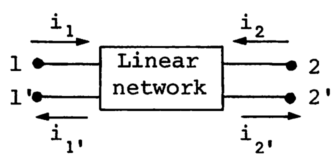
Impedance parameters are defined by the following two sets of equations.

$$v_1 = z_{11} i_1 + z_{12} i_2$$

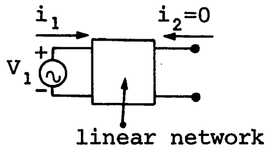
$$v_2 = z_{21} i_1 + z_{22} i_2$$

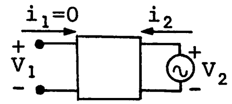
where v_1 and v_2 are acting as independent variables.

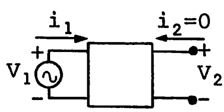
(Note: A general linear two-port network is being considered.)

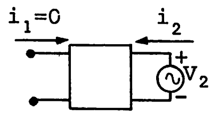


By setting i_1 and i_2 equal to zero, four impedance parameters are defined as follows:

<table>
<tr><th>$i_2=0$</th><th>$i_1=0$</th></tr>
<tr><td>

Open-circuit input impedance</td><td>

$Z_{12} = \frac{V_1}{i_2}$ (Ω)
Open-circuit reverse transfer impedance</td></tr>
<tr><td>

Open-circuit forward transfer impedance</td><td>

$Z_{22} = \frac{V_2}{i_2}$ (Ω)
Open-circuit output impedance</td></tr>
</table>

(Note: Since i_1 and i_2 are set equal to 0, Z-parameters are also called open-circuit impedance parameters.)

12.2 HYBRID - PARAMETERS

12.2.1 HYBRID OR H-PARAMETERS

The usage of hybrid parameters is for the analysis of transistor circuits. The hybrid parameters are defined by two sets of equations as follows:

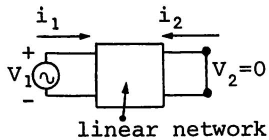
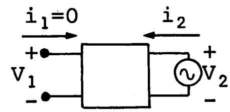
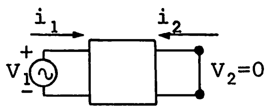
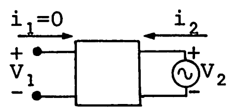
$$v_1 = h_{11}I_1 + h_{12}v_2$$

$$I_2 = h_{21}I_1 + h_{22}v_2$$

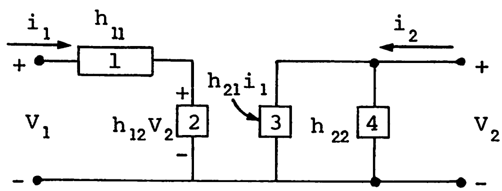
where v_1 and I_2 are acting as independent variables.

The hybrid parameters are determined by the use of short-circuit and open-circuit conditions.

Hence,

$V_2=0$	$I_2=0$
 $h_{11} = \frac{V_1}{I_1}$ (Ω) ↓ Short-circuit input impedance	 $h_{12} = \frac{V_1}{V_2}$ ↓ Open-circuit reverse voltage ratio (gain)
 $h_{21} = \frac{I_2}{I_1}$ Short-circuit forward current ratio (gain)	 $h_{22} = \frac{I_2}{V_2}$ $(\mho)$ ↓ Open-circuit output admittance

The hybrid parameters in a two-port network are defined as shown in the network below.

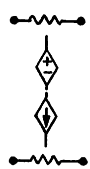


where $\boxed{1}$ = resistance (Ω)

$\boxed{2}$ = dependent voltage source

$\boxed{3}$ = dependent current source

$\boxed{4}$ = conductance $(\mho)$



12.3 ADMITTANCE PARAMETERS

12.3.1 ADMITTANCE OR Y-PARAMETERS

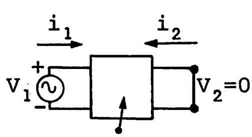
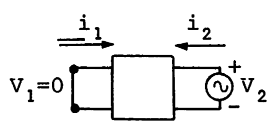
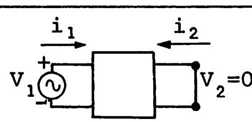
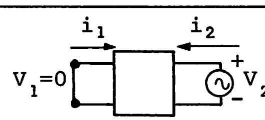
Admittance parameters are described by the following two sets of equations:

$$i_1 = y_{11}v_1 + y_{12}v_2$$

$$i_2 = y_{21}v_1 + y_{22}v_2$$

Then each parameter is defined by setting v_1 or v_2 equal to zero. Hence, Y-parameters are also called the short-circuit admittance parameters.

Thus, by setting:

$V_2=0$	
linear network $y_{11} = \frac{i_1}{V_1}$ Short-circuit input admittance	$y_{12} = \frac{i_1}{V_2}$ Short-circuit reverse transfer admittance
$y_{21} = \frac{i_2}{V_1}$ Short-circuit forward transfer admittance	$y_{22} = \frac{i_2}{V_2}$ Short-circuit output admittance

12.4 Z,Y, AND H PARAMETERS AND RELATIONSHIPS

Let us define the following matrices:

$$[Z] = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}, \quad [Y] = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}, \quad [H] = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

then, $\Delta z = Z_{11} Z_{22} - z_{12} z_{21}$, $\Delta y = Y_{11} Y_{22} - y_{12} y_{21}$, $\Delta h = h_{11} h_{22} - h_{12} h_{21}$

are the determinants of [Z], [Y] and [H] respectively.

Now, the conversion between parameters are defined as follows:

(1A) $Z \rightarrow Y \implies \begin{bmatrix} \dfrac{z_{22}}{\Delta z} & \dfrac{-z_{12}}{\Delta z} \\ \dfrac{-z_{21}}{\Delta z} & \dfrac{z_{11}}{\Delta z} \end{bmatrix}$ (1B) $Z \rightarrow H \implies \begin{bmatrix} \dfrac{\Delta z}{z_{22}} & \dfrac{z_{12}}{z_{22}} \\ \dfrac{-z_{21}}{z_{22}} & \dfrac{1}{z_{22}} \end{bmatrix}$

(2A) $Y \rightarrow Z \implies \begin{bmatrix} \dfrac{y_{22}}{\Delta y} & \dfrac{-y_{12}}{\Delta y} \\ \dfrac{-y_{21}}{\Delta y} & \dfrac{y_{11}}{\Delta y} \end{bmatrix}$ (2B) $Y \rightarrow H \implies \begin{bmatrix} \dfrac{1}{y_{11}} & \dfrac{-y_{11}}{y_{11}} \\ \dfrac{y_{21}}{y_{11}} & \dfrac{\Delta y}{y_{11}} \end{bmatrix}$

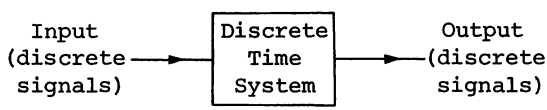
(3A) $H \rightarrow Z \implies \begin{bmatrix} \dfrac{\Delta h}{h_{22}} & \dfrac{h_{12}}{h_{22}} \\ \dfrac{-h_{21}}{h_{22}} & \dfrac{1}{h_{22}} \end{bmatrix}$ (3B) $H \rightarrow Y \implies \begin{bmatrix} \dfrac{1}{h_{11}} & \dfrac{-h_{12}}{h_{11}} \\ \dfrac{h_{21}}{h_{11}} & \dfrac{\Delta h}{h_{11}} \end{bmatrix}$

CHAPTER 13

DISCRETE SYSTEMS AND Z - TRANSFORMS

13.1 DISCRETE - TIME SYSTEMS

13.1.1 A DISCRETE-TIME SYSTEM

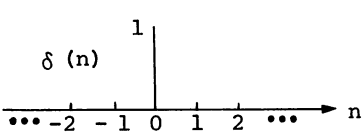


Since discrete signals may be represented by a sequence of numbers, knowing the characteristics of such a sequence is essential.

The characteristics of some general sequences are listed below:

1. Kronecker Delta Sequence:

$$\delta(n) = \begin{cases} 1, & n=0 \\ 0, & n=\pm1,\pm2,\ldots \end{cases}$$

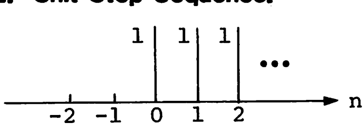


$$\delta(n-i) = \begin{cases} 1, & n=i \\ 0, & \text{elsewhere} \end{cases}$$

(Note: i is an arbitrary integer.)

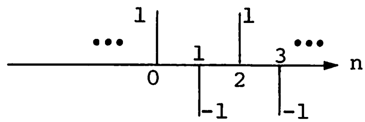
2. Unit Step Sequence:

$$u(n) = \begin{cases} 0, & n=-1,-2,-3\ldots \\ 1, & n=0,1,2,\ldots \end{cases}$$



3. Unit Alternating Sequence:

$$u(n) = \begin{cases} 0, & n=-1,-2,-3\ldots \\ (-1)^n, & n = 0,1,2,\ldots \end{cases}$$



4. Unit Ramp Sequence:

$$u(n) = \begin{cases} 0, & n=-1,-2,\ldots \\ n, & n=0,1,2,\ldots \end{cases}$$

13.2 FIRST - ORDER LINEAR DISCRETE SYSTEM

A first-order linear discrete system is represented by the linear first-order difference equation as follows:

$$y(n) + A_1 y(n-1) = B_0 u(n) + B_1 u(n-1) \tag{1}$$

where u and y are denoted as the input and output of the system, respectively.

If the input signal is applied at n = 0, then eq.(1) becomes

$$y(0) = B_0 u(0) + B_1 u(-1) - A_1 y(-1)$$

where y(-1) is the initial condition of the system.

(Note: $u(-1) = 0$)

In general, where an input signal is applied to n = j, then the response of the system is represented as follows:

$$y(j) = B_0 u(j) + B_1 u(j-1) - A_1 y(j-1)$$

Since $B_1 u(j-1) = 0$

hence, $y(j) = B_0 u(j) - A_1 y(j-1)$

(Note: The initial condition is defined by $y(j-1)$.)

Summary:

In general, a linear discrete system is described by the relationship as follows:

$$y(n) = B_0 u(n) + B_1 u(n-1) + \ldots + B_i u(n-i) - A_1 y(n-1)$$
$$- A_2 y(n-1) - \ldots - A_N y(n-N)$$

where $\left.\begin{array}{l} B_0 \ldots B_i \\ A_1 \ldots A_N \end{array}\right. = \text{constants}$

and i and N = fixed non-negative integers.

If an input signal is applied at $n = j$, then

$$y(j) = B_0 u(j) - A_1 y(j-1) - A_2 y(j-2) - \ldots - A_N y(j-N)$$

(Note: $u(j-i) = 0$ where $i = 1,2,3,\ldots.$)

13.3 CLOSED-FORM IDENTITIES

Closed-form identity is useful in expressing the response of a linear system.

Some generally used closed-form identities are given below:

1) $$\sum_{m=0}^{N} r^m = \frac{1 - r^{N+1}}{1 - r} \qquad \text{where } r \neq 1$$

2) $$\sum_{m=0}^{N} m r^m = \frac{r}{(1-r)^2}\left[1 - r^m - m r^m + m r^{m+1}\right] \qquad \text{where } r \neq 1$$

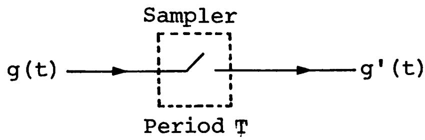
3) $$\sum_{m=0}^{N} m^2 r^m = \frac{r}{(1-r)^3}\left[(1+r)(1-r^N)-2(1-r)Nr^N-(1-r)^2N^2r^N\right]$$

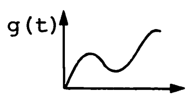
where $r \neq 1$

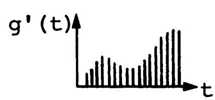
13.4 THE Z - TRANSFORM

When a continuous function of time g(t) is sampled at regular intervals of period T, the usual Laplace transform techniques are modified.

The diagrammatic form of a simple sampler together with its associated input-output waveforms is shown below.







Note: The sampling frequency $\equiv f_s = \frac{1}{T}$

Defining the set of impulse function $\delta_T(t)$ by

$$\delta_T(t) \equiv \sum_{n=0}^{\infty} \delta(t-nT)$$

the input-output relationship of the sampler becomes

$$g'(t) = g(t) \cdot \delta_T(t)$$

$$= \sum_{n=0}^{\infty} g(nT) \cdot \delta(t-nT).$$

Note: For a given g(t) and T, the function g'(t) is unique. However the converse is not true.

The variable 'z' is introduced by means of the transformation:

$$z = e^{Ts}$$

and since any function of s can now be replaced by a corresponding function of z, we have

$$\boxed{G(z) = \sum_{n=0}^{\infty} g(nT) \cdot z^{-n}}$$

where $G'(s) \equiv G(z)$

and $s = \frac{1}{T} \ln z$

The z operator can now be defined in terms of the Laplace operator by the relationship

$$Z\{g(t)\} \equiv L\{g'(t)\}$$

or

$$Z\{g(t)\} = \Sigma \text{residues of } \left[\left(\frac{1}{1-e^{Tx}z^{-1}}\right) G(z)\right]$$

The inverse z transform is

$$Z^{-1}\{G(z)\} \equiv g'(t)$$
$$= \frac{1}{2\pi j} \oint G(z) \cdot z^{n-1}\,dz$$

where the contour of integration encloses all the singularities of the integrand.

13.5 PROPERTIES OF Z - TRANSFORM

$g(t)$	$Z\{g(t)\} = G(z)$
1. Linearity: $Af(t)+Bg(t)$	$AF(z)+BG(z)$
2. Left shifting: $g(t+mT)$	$z^m G(z) - \sum_{r=0}^{m-1} z^{m-1} g(rT)$ $= z^m G(z)$ when $g(rT) = 0$, $0 \le r \le m-1$
3. Right shifting: $g(t-mT)$	$z^{-m} G(z)$
4. Summation: $\sum_{m=0}^{T/t} g(mT)$	$\left(\frac{z}{z-1}\right) G(z)$
5. Differentiating: $tg(t)$	$-Tz \frac{d}{dz} G(z)$
6. Integrating: $t^{-1}g(t)$	$-\frac{1}{T} \int_0^{z} \frac{G(z)}{z} dz$
7. Convolution: $\sum_{r=0}^{t} g_1(t-r)g_2(r)$	$G_1(z)G_2(z)$

8. Initial value Theorem

$$g(0) = \lim_{|Z|\to\infty} G(z)$$

9. Final value Theorem:

$$g(\infty) = \lim_{z\to 1} (z-1)G(z)$$

if $(z-1)G(z)$ is analytic for $|z| \geq 1$.

13.6 METHODS OF EVALUATING INVERSE Z - TRANSFORMS

1) Cauchy's residue theorem;

For $t = nT$,

$$g(nT) = \sum_{\text{all } z_k} [\text{residues of } G(z)z^{n-1} \text{ at } z_k]$$

where z_k defines all of the poles of $G(z)z^{n-1}$

2) Partial fractions:

Expand $\frac{G(z)}{z}$ into partial fractions. The product of z with each of the partial fractions will then be recognizable from the standard forms in the table of z transforms. Note however that the continuous functions obtained are only valid at the sampling instants.

3) Power series expansion by long division using detached coefficients:

$G(z)$ is expanded into a power series in z^{-1} and the coefficient of the term in z^{-n} is the value of $g(nT)$. i.e., the value of $g(t)$ at the nth sampling instant.

13.6.1 THE Z TRANSFORM AS A MEANS OF DETERMINING APPROXIMATELY THE INVERSE LAPLACE TRANSFORM

Since $Z = e^{Ts}$

$$S^{-1} = \frac{T}{2}\left[\frac{1}{v} - \frac{v}{3} - \frac{4v^3}{45} - \frac{44v^5}{945} - \ldots\right]$$

where

$$v \equiv \frac{1 - z^{-1}}{1 + z^{-1}},$$

the series being very rapid in its convergence. Given $G(s)$ to find its inverse Laplace transform, the following operations are carried out:

1) Divide the numerator and denominator of $G(s)$ by the highest power of s, yielding as an alternate form for $G(s)$ (the quotient of two polynomials in s^{-1}).

2) Choose as a numerical value of T, which makes $2\pi/T$ much larger than the imaginary part of the poles of $G(s)$.

3) Substitute the alternative form for $G(s)$ obtained in (1) above; the expansion for s^{-n} can be determined from the following short table of approximations.

   Do not, at this stage, insert the numerical value for T because tabulations with different intervals may be required.

4) Divide by T.

5) Insert the chosen value for T and divide the numerator by the denominator.

6) The coefficient of z^{-n} is the required value of the function at $t = nT$.

S^{-n}	Z-transform (approximation)
S^{-1}	$\frac{T}{2}\left[\frac{1 + Z^{-1}}{1 - Z^{-1}}\right]$
S^{-2}	$\frac{T^2}{12}\left[\frac{1 + 10Z^{-1} + Z^{-2}}{(1 - Z^{-1})^2}\right]$
S^{-3}	$\frac{T^3}{3}\left[\frac{Z^{-1} + Z^{-2}}{(1 - Z^{-1})^3}\right]$
S^{-4}	$\frac{T^4}{144}\left[\frac{1 + 20Z^{-1} + 102Z^{-2} + 20Z^{-3} + Z^{-4}}{(1 - Z^{-1})^4}\right]$
S^{-5}	$\frac{T^5}{124}\left[\frac{Z^{-1} + 11Z^{-2} + 11Z^{-3} + Z^{-4}}{(1 - Z^{-1})^4}\right]$
S^{-6}	$\frac{T^6}{4}\left[\frac{Z^{-2} + 2Z^{-3} + Z^{-4}}{(1 - Z^{-1})^6}\right]$
S^{-7}	$\frac{T^7}{8}\left[\frac{Z^{-2} + 3Z^{-3} + 3Z^{-4} + Z^{-5}}{(1 - Z^{-1})^7}\right]$

13.7 Z - TRANSFORM PAIRS

Transform Pair Signal	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1 - Z^{-1}}$	$\lvert Z\rvert > 1$
$u[-n - 1]$	$\frac{1}{1 - Z^{-1}}$	$\lvert Z\rvert < 1$
$\delta[n - m]$	Z^{-m}	All Z except 0 (if $m > 0$) or ∞ (if $m < 0$)
$\alpha^n u[n]$	$\frac{1}{1 - \alpha Z^{-1}}$	$\lvert Z\rvert > \lvert\alpha\rvert$
$-\alpha^n u[-n - 1]$	$\frac{1}{1 - \alpha Z^{-1}}$	$\lvert Z\rvert < \lvert\alpha\rvert$
$n\alpha^n u[n]$	$\frac{\alpha Z^{-1}}{1 - \alpha Z^{-1})^2}$	$\lvert Z\rvert > \lvert\alpha\rvert$
$-n\alpha^n u[-n - 1]$	$\frac{\alpha Z^{-1}}{(1 - \alpha Z^{-1})^2}$	$\lvert Z\rvert < \lvert\alpha\rvert$
$[\cos\omega_0 n]\ u[n]$	$\frac{1-[\cos\omega_0]Z^{-1}}{1-[2\cos\omega_0]Z^{-1}+Z^{-2}}$	$\lvert Z\rvert > 1$
$[\sin\omega_0 n]\ u[n]$	$\frac{[\sin\omega_0]Z^{-1}}{1-[2\cos\omega_0]Z^{-1}+Z^{-2}}$	$\lvert Z\rvert > 1$

CHAPTER 14

TOPOLOGICAL ANALYSIS

14.1 INCIDENT MATRIX

By definition, an augmented incident matrix Aa, is an n(nodes) × b(branches) matrix of a directed graph of any planar network, i.e.,

$$Aa = [a_{ij}]_{n \times b}$$

where

$$a_{ij} = \begin{cases} 1 & \text{when branch bj is incident to node } n_i \text{ and the reference current, } i_j\text{, leaves the node.} \\ -1 & \text{when branch bj is incident to node } n_j \text{ and the reference current, } i_j\text{, enters the node.} \\ 0 & \text{when branch bj is not incident to node } n_i. \end{cases}$$

The incident matrix Aa can be represented as:

$$Aa = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_i \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & \dots & b_j \end{array} \\ \left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right] \end{array}$$

Note: An incidence submatrix can be obtained by taking out any one of the rows of the incidence matrix Aa.

14.2 THE CIRCUIT (LOOP) MATRIX

By definition, an augmented circuit matrix, B_a, is an $\ell \times b$ matrix where ℓ = loops and b = branches.

Hence, $Ba = [b_{ij}]_{\ell \times b}$

where

$$b_{ij} = \begin{cases} 1 & \text{when branch bj is in loop } \ell_i \text{ and is oriented in the same direction.} \\ -1 & \text{when branch bj is in loop } \ell_i \text{ and is oriented in the opposite direction.} \\ 0 & \text{when branch bj is not in loop } \ell_i. \end{cases}$$

The circuit matrix can be represented as:

$$Ba = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \ell_3 \\ \vdots \\ \ell_i \end{array} \begin{array}{c} b_1 \; b_2 \; b_3 \ldots b_j \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array} \right] \end{array}$$

14.3 FUNDAMENTAL (LOOP) MATRIX

The fundamental loop (circuit) matrix, B_f, is defined as a $[b-(n-1)] \times b$ matrix where b = branches and n = nodes. i.e.,

$$B_f = [b_{ij}]_{[b-(n-1)] \times b}$$

where

$$b_{ij} = \begin{cases} 1 & \text{when branch bj is in the fundamental loop } \ell_i \text{ and is oriented in the same direction.} \\ -1 & \text{when branch bj is in the fundamental loop } \ell_i \text{ and is oriented in the opposite direction.} \\ 0 & \text{when branch bj is not in the fundamental loop } \ell_i. \end{cases}$$

Note: A fundamental loop cannot contain more than one chord; a chord is any branch of a cotree and a cotree is the set of all branches not in a tree. (A tree is defined in Chapter 5.)

The matrix representation of B_f is as follows:

$$B_f = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \ell_3 \\ \vdots \\ \ell_i \\ \vdots \\ \ell_{b-(n-1)} \end{array} \begin{array}{c} b_1 \; b_2 \; b_3 \; \ldots \; b_j \ldots \; b_b \\ \left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array} \right] \end{array}$$

CHAPTER 15

NUMERICAL METHODS

15.1 NEWTON'S METHOD

Newton's method is used to find the roots of a polynomial equation.

Consider the equation:

$$F(s) = k_4 s^4 + k_3 s^3 + k_2 s^2 + k_1 s^1 + k_0 = 0.$$

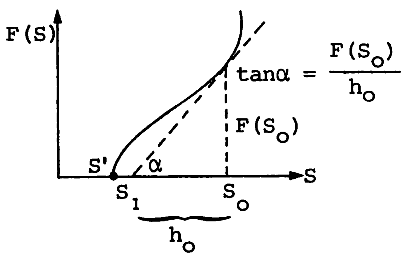
In order to determine the real value of $s = s'$ such that $F(s) = 0$, the Newton's method is used as follows:

Since all the coefficients of $F(s)$ are of the same sign and real, the complex roots of $F(s)$ are complex conjugate pairs and the real roots, if any, are negative.

Now, by inspection, we begin with a guess, $s = s_0$, for the root. Also let $s_1 = s_0 - h_0$ where

s_1 = a closer approximation of the root obtained from s_0 and

$$h_0 = s_0 - s_1 = \frac{-F(s_0)}{F'(s_0)} \quad \text{and} \quad F'(s_0) = \frac{d}{ds} F(s)\bigg|_{s=s_0}$$



Then, in general,

$$S_{i+1} = S_i - \frac{F(s_i)}{F'(s_i)} \quad = (i+1) = \text{iteration of the initial approximation } s_0.$$

i.e., S_i = the previous approximation (or guess) of the root.

and S_{i+1} = the new approximation of $F'(S_i)$, which is the derivative of $F(S)$ evaluated at $S = S_i$.

Finally, the iteration is stopped when S_{i+1} is approximately equal to S_i, indicating that the value of S_i is the root s'.

15.2 SIMPSON'S RULE

Simpson's rule states that

$$\int_{x_0=a}^{x=b} y(x)dx \cong \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$$

where $h = \frac{b-a}{n}$ (note: n is even.)

and $y_0 = y(a)$, $y_1 = y(a+h)$, $y_2 = y(a+2h)$, $y_n = y(a+nb) = y(b)$

Note: The more values between the limits of integration taken, (the larger n is), the more accurate the result will be.

Simpson's rule is a simple and reasonably accurate method which can be used to write programs for digital computers or programmable calculators. The flow chart shown below is for Simpson's rule of integration.