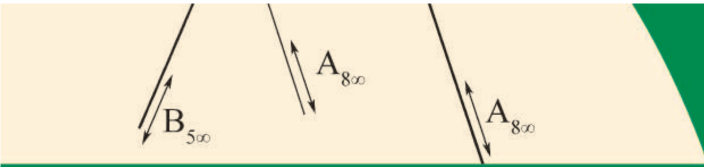
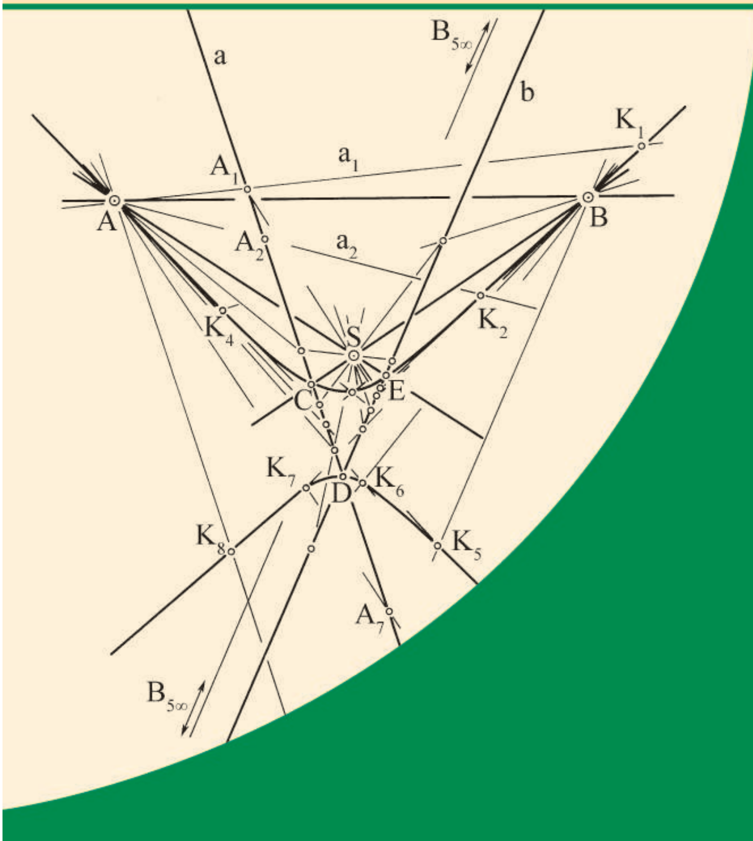


Robert Neumann (Ed.)

Topics in Mathematics

for the 11th Grade

Based on teaching practices in Waldorf schools



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Pädagogische
Forschungsstelle
Kassel

Topics in Mathematics

for the 11th Grade

Based on teaching practices in Waldorf schools

Analytical Geometry
Projective Geometry
Spherical Geometry
Sequences and Limits

Peter Baum, Karl-F. Georg, Uwe Hansen, Markus Hünig,
Klaus Labudde, Rolf Rosbigalle, Stephan Sigler
Editor: Robert Neumann

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Preface

With this volume, Topics in Mathematics for the 11th Grade, the «Math Curriculum Initiative» work-group presents its most substantial production so far.

The focal topic being projective geometry, it seemed reasonable to go into considerable detail here, especially since there is hardly any literature about teaching this topic. The few – but precious – existing texts are referenced in the articles. Aside from this it was a concern of the team to present a broad cross-section of the great variety of possible ways of handling the subject matter especially of this topic in the classroom. Thus one will find very diverse approaches in the various articles devoted to introducing projective geometry in the 11th Grade. They have resulted from the particular teaching situations as well as the personal experiences and leanings of the authors. They represent a broad spectrum of alternatives which should stimulate the reader ultimately to find his or her own way.

In contrast, spherical geometry and trigonometry are frequently dealt with in the technical literature, whether in relation to geodesy, cartography or astronomy, so the emphasis here was on the choice of subject matter and its method of preparation.

Analytic geometry is treated in many textbooks. For instruction in a Waldorf school many considerations arise relating to content or methodology for which there is no provision in these books, however. Thus, even those familiar with analytic geometry as customarily treated in schools will be able to discover in this volume relevant suggestions, interesting cross connections and unexpected approaches.

A few articles containing background knowledge, little insights, or basic pedagogic observations round off the book.

The authors, who have brought together their experiences with such commitment, hope that this book, too, will be able to provide beginners with assistance in getting started and experienced colleagues with stimuli for further work.

I would like to express our warm thanks to Sebastian Labusch, who did a great deal of the word-processing and type-setting.

Markus Hünig Mülheim an der Ruhr, Spring 2006

Preface to the English Edition

This book is the second in the series of «green books» for mathematics teachers to be translated. It is considerably more extensive than the tenth-grade book, chiefly due to its many articles about projective geometry. Because many teachers may be relatively unfamiliar with this topic, we decided to translate all of these. Each is presented according to its author's unique viewpoint, providing a good opportunity to become acquainted with the topic.

Projective geometry is rarely taught outside of Waldorf Schools. As it offers a wonderful opportunity to exercise the imagination and thinking, extending these to embrace the points at infinity without leaving the secure ground of mathematics, it would be advantageous for this branch of geometry to be more widely taught.

As the book was translated by several different people, the articles might vary at times in expressions and style.

Special thanks go to the translators, Paul Courtney, Brent Daeuble, Harlan Gilbert and Charles Gunn, and to those who funded the translation, the Pädagogische Forschungsstelle in Kassel and the Association of Waldorf Schools of North America.

Robert Neumann

December 2010

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Introduction to analytical geometry – a possible start for a main lesson block

KARL-FRIEDRICH GEORG

With the introduction to analytical geometry, the goal is that certain geometric questions be made accessible to computational treatment in the most simple and broad way possible. For this, we will begin with the classical coordinate method. The traditional construction first treats the geometry of linear structures (linear algebra) – later, this is continued with the geometry of structures of the second order. This also ties to the intensification of the algebraic demands. Today, the lesson plans of the cultural ministries include mainly «linear algebra» under analytic geometry- and, as non-plane structures, the cube at best. In comparison to the treatment of conic sections, a strong reduction to formal algebra, to the detriment of a visual geometric endeavor, is connected with this. The skills of practical construction are no longer taught.

If analytical geometry is called for in the Waldorf School's eleventh grade lesson plans, up to the conic sections, it is built on diverse experience of these students with curves: In the ninth grade, the conic sections were introduced as curves of position. In the tenth grade, these conic sections were constructed as intersecting lines of cone and plane. The following step in the eleventh grade shows that these known curves can also be grasped algebraically: an equation in connection with a coordinate system represents a curve and the reverse is true. The place definition of a curve leads to its characteristic equation. Here we are aware that the fundamentals have geometric content.

This main-lesson block clarifies the changing relation of equation and curve – without having to concern oneself just with distances, line equations, and partial ratios. Such an entry into this block can be carried out in the following way. First of all, the student experiences the direct relationship of equation and attendant form (curve). The curves are either known or are constructed additionally. Secondly, for setting up the value tables, an exercise in the practical use of the calculator is provided.

A continuation of the study of curves can take place in the twelfth grade – in the free geometry of plane curves.

The following explanation corresponds to the beginning of a block.

In the further course of the block, for the themes of distance and line, the areas are selected which form the basis for being able to master derivation and problems with conic sections, circle, ellipse, hyperbola, and parabola. When the student achieves confidence with calculation in regard to circles (pole/polar) , we need less time with the other curves. If we are not to shortchange the element of construction, a block length of four weeks is necessary.

Introduction

For the equation $x^2 + y^2 = 25$ there is an infinite number of number pairs (x, y) which fulfills it. For purposes of calculation, we have solved this equation for y and made a table

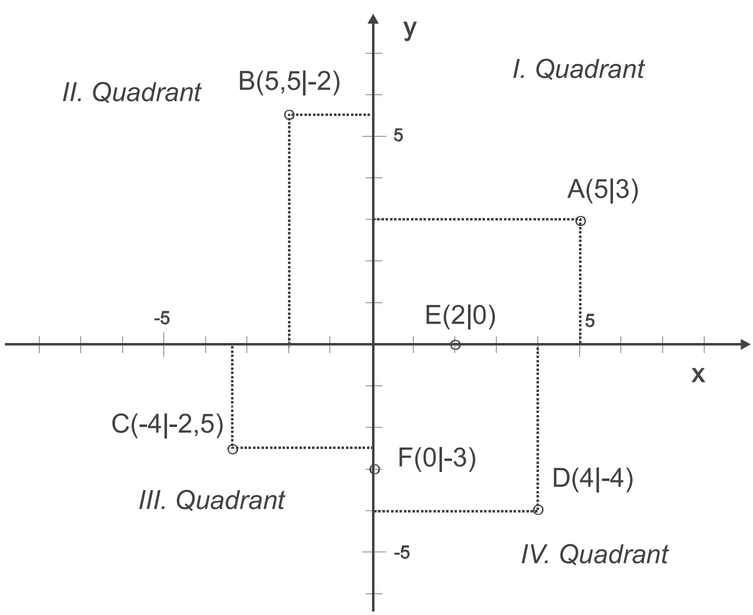
$$y^2 = 25 - x^2; \quad y = \pm\sqrt{25 - x^2}$$

x	3	3	4	0	5	-3	-4	1	2	-1	-2
y	4	-4	± 3	± 5	0	± 4	± 3	± 4.90	± 4.58	± 4.90	± 4.58

It is clear from the number pairs that this equation conveys a certain relation between the unknowns x and y. Both unknowns can take on different values, but not arbitrarily – rather, they are dependent on each other. Thus, in such equations, we call x and y variables and the equation relationship equation.

In a coordinate system, we can visualize each numeric pair (x, y) as a point. The horizontal axis is the x-axis and the vertical is the y-axis (see below). The intersecting point of both axes is the origin of coordinates.

We call x the abscissa and y the ordinate of the point P. Each number pair arranged (x, y) represents a point $\mathrm{P}(x, y)$; we call x and y the coordinates of the point P. We also call the crossing of axes standing vertically to each other a Cartesian coordinate system, named after one of the two founders of analytic geometry, Rene Descartes. He was called Cartesius (1596-1650). We can also visualize the coordinates of the point P as distances on the coordinate axes or on lines parallel to it.



We enter the number pairs of our table into the coordinate system as points.

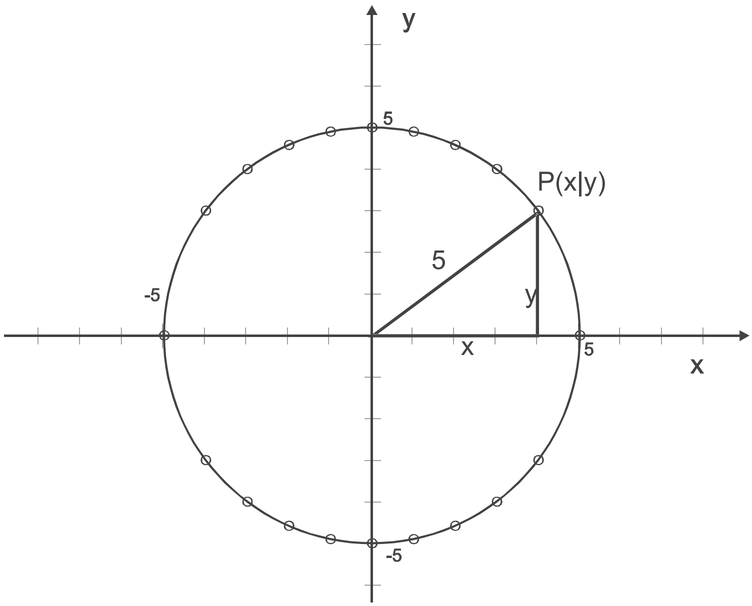


Fig. 2

Since with each point, x and y form the sides of a right angle triangle and the hypotenuse remains a constant 5, all points lie on a curve around the origin. The equation $y = \sqrt{25 - x^2}$ only makes sense when $25 - x^2 \geqslant 0$, that ist $25 \geqslant x^2$.

This is only achieved when $x \leqslant 5$ and $x \geqslant -5$, as well as: $-5 \leqslant x \leqslant 5$.

In the analytical geometry of the plane, we are dealing with equations in which two variables x and y occur. The number of all solutions (x, y) of such equations represents the number of all points $\mathrm{P}(x, y)$of straight lines or curves. Accordingly, there are two basic problems:

1. The determination of the geometric properties of a line, which is provided by an equation.
2. The formation of the equation of a line, proceeding from its geometric properties.

Our introductory example is an example for the basic problem.

Further examples:

Example 1

$16x - 3y^2 = 0$; solving for y, the equation reads: $y = \pm\frac{4}{3}\sqrt{3x}$.

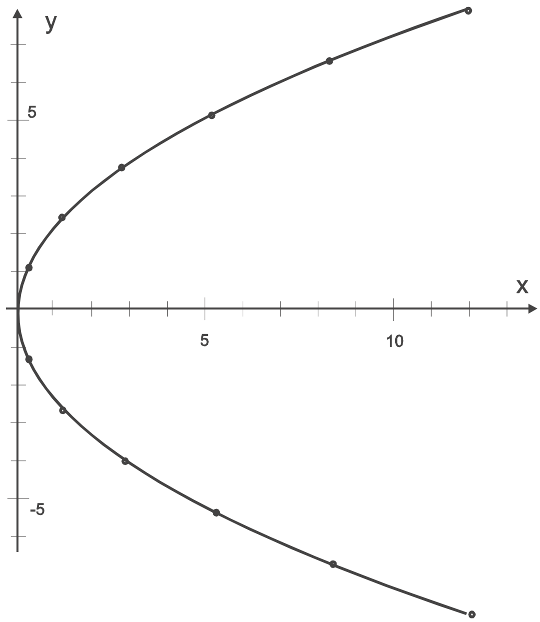
For x, we enter some suitable values and enter the solutions in the table:

x	0	$\frac{1}{3}$	$\frac{4}{3}$	3	$\frac{16}{3}$	$\frac{25}{3}$	12
y	0	$\pm\frac{4}{3}$	$\pm\frac{8}{3}$	± 4	$\pm\frac{16}{3}$	$\pm\frac{20}{3}$	± 8

The resulting curve could be a parabola.

From the equation, we conclude the following:

There are no points with negative abscissa which lie in a curve. For every positive x-value, there are two different y-values, which distinguish themselves only by the sign. So the curve is symmetric in regard to the x-axis. As x increases, the value of y grows – so the curve becomes increasingly distant from the x-axis.



Example 2

$x^2 - y^2 = 4$; for the solution of y, we get this result: $y = \pm\sqrt{x^2 - 4}$.

Here we also set up a table with examples of solution pairs:

x	± 2	± 2.5	± 3	± 4	± 5	± 6	± 7
y	0	± 1.5	± 2.24	± 3.64	± 4.58	± 5.66	± 6.71

... and we enter the number pairs as points.

With this example, a curve results, which could be a hyperbola. From the equation, we conclude: there are no points for $-2 < x < 2$, which lie on the curve.

Also here there are two different y-values for each x-value available – the only difference is the sign. So the curve is symmetric in regard to the x-axis. Since, for the positive and negative x-values, the same two y-values result, the curve is also symmetric to the y-axis.

Projective Geometry – the Main Lesson Block in the 11th Grade

KARL-FRIEDRICH GEORG

Preliminary Remarks

In addition to Rudolf Steiner's curriculum specifications for the eleventh grade, a projective geometry main lesson has found its place in Waldorf schools, generally as a replacement for the spherical geometry that was originally proposed. Through this augmenting of the Euclidean thought-world, relationships can be discovered that are not given merely through sense perception, and that only unfold through the intellect, through thinking. Through the synthetic approach and using the elements of geometry the relationships of position (incidence) are grasped and put into practice independent of any metric, the point being to bring the relationships into movement in the imagination. On this basis, central projections can be handled metrically in the 13th grade (when no *Zentralabitur* (central final examination) is prescribed). By means of challenging exercises (both computational and constructive) students can apply these projections in many different situations.

In the years since Rudolf Steiner gave new impulses to mathematicians to establish projective geometry as the basis for a new way of thinking in natural scientific research, this work has been carried out notably by George Adams (*Strahlende Weltgestaltung*, 1933), Louis Locher-Ernst (*Projektive Geometrie*, 1944; *Raum und Gegenraum*, 1957) and, especially as a basis for teaching, by Arnold Bernhard (*Projektive Geometrie*, 1984).

Experience over the span of three decades has shown, in a variety of ways, how important this main lesson is. Although the concepts of projective geometry are not difficult, they are in their very nature altogether new and require a way of thinking which is not given either by the Euclidean way or by the analytic method, namely a qualitative understanding of mathematical form. So students get to exercise forms of thinking that challenge them and develop them in a way that the ordinary mathematical curriculum can never do.

The following exposition corresponds to the content and sequence of a four-week main lesson. In some parts I have oriented myself to the book by Arnold Bernhard which has an immense wealth of material. I hope that this description of a main lesson block can be a useful aid, particularly for beginning teachers.

God establishes himself without reason and measures himself without measurement.
Should you be at one with him in spirit, O man, you will understand.
Angelus Silesius (1624-1677)

The actions of space and time are creation's powers,
and their relationships are the hinges of the world.
Novalis (1772-1801)

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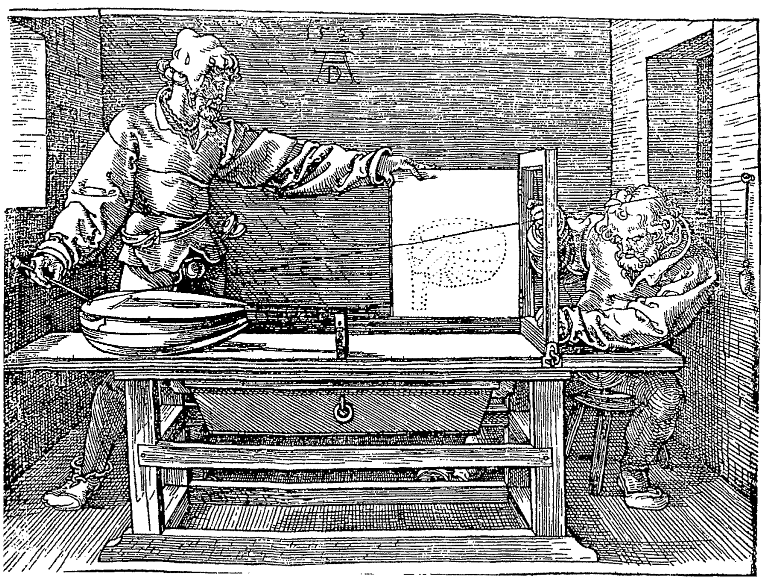
What is grasped in thought is also effective in the world.
What lives in the world also comes to be revealed in thought.
Louis Locher-Ernst (1906-1962)

1 Central projection

In the past the study of geometry was concerned with figures of finite magnitude. But since about the Renaissance, geometers and artists have increasingly occupied themselves with geometric inquiries that broke the bounds of the finite. Painters endeavored to portray buildings and other forms in space so that the spatial impression of the image agrees with the impression of sensory reality. They solved this problem with the help of perspective. We'll see that perspective is nothing other than a central projection. In a drawing, *Albrecht Dürer* vividly portrayed the technique of central projection:



From: Albrecht Dürer, *Underweysung der Messung mit dem Zirkel und dem Richtscheyt* [*Instruction in measurement with compasses and straight-edge*], Nürnberg, 1525.

A thread is stretched from a center (the ring on the wall) to each point of an object that is to be depicted. Between the center and the object is placed an image plane. Where the thread intersects the image plane, a point is marked: it is the image point of the point on the object. If we bring our eye to where the center is, we see each image point in the same direction as its object point. In geometric language: the set of object points (in general located in space) and the set of image points (always lying in a plane) are perspectively related; an image point and its object point lie in the same projection ray through the center. This way of forming an image is called *central projection*.

Examples of the use of central projection are photography and slide projection.

1. Problem: Project the given triangle *ABC* lying in the plane ε into the horizontal plane $\overline{\varepsilon}$ (Figure 1.1)

2. Problem: Project the triangle *ABC* from *Z* onto the horizontal plane $\overline{\varepsilon}$ (Figure 1.2)

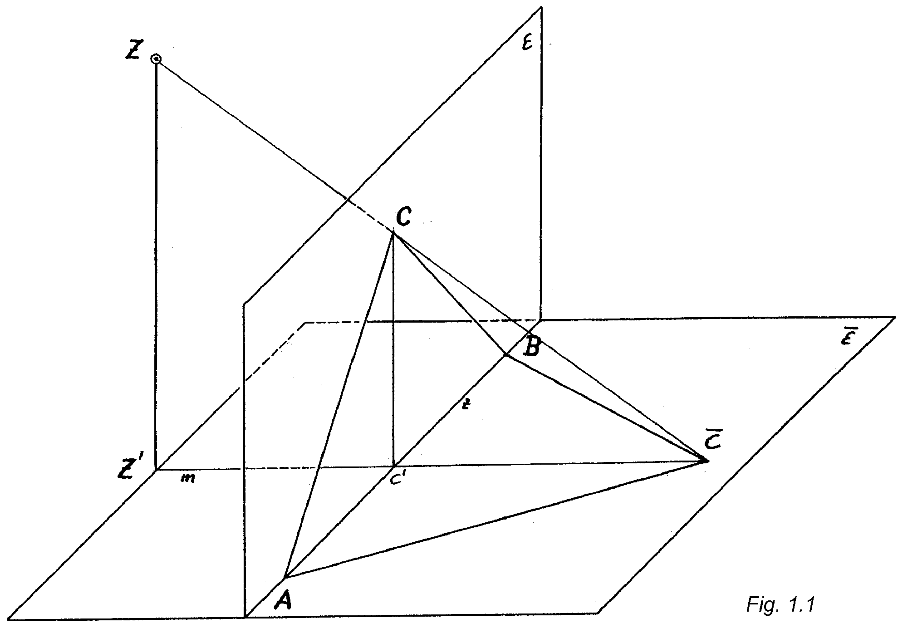


Fig. 1.1

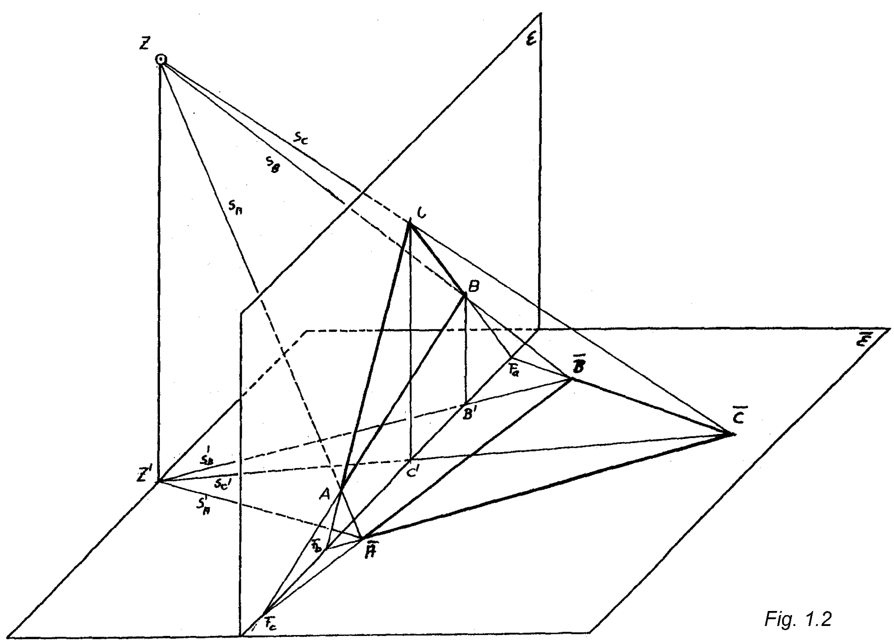


Fig. 1.2

The extended sides a, b, c of the triangle in ε and their images $\overline{a}$, $\overline{b}$, $\overline{c}$ in $\overline{\varepsilon}$ intersect in the «axis» z. Whereas each point of the line a is projected onto a corresponding image point in $\overline{a}$, the common point of a and $\overline{a}$ in the axis z is the only point of a that is projected onto itself. It is called the «fixed point» of the line a.

3. Problem: The pyramid *ABCS* (*S* vertically above *B*) stands on a table top. Project it from *Z* onto the «wall». (Figure 1.3)
4. Problem: Project the triangle ABC from *Z* onto the horizontal plane and then let the «candle» Z_1Z' burn down. (Figure 1.4)

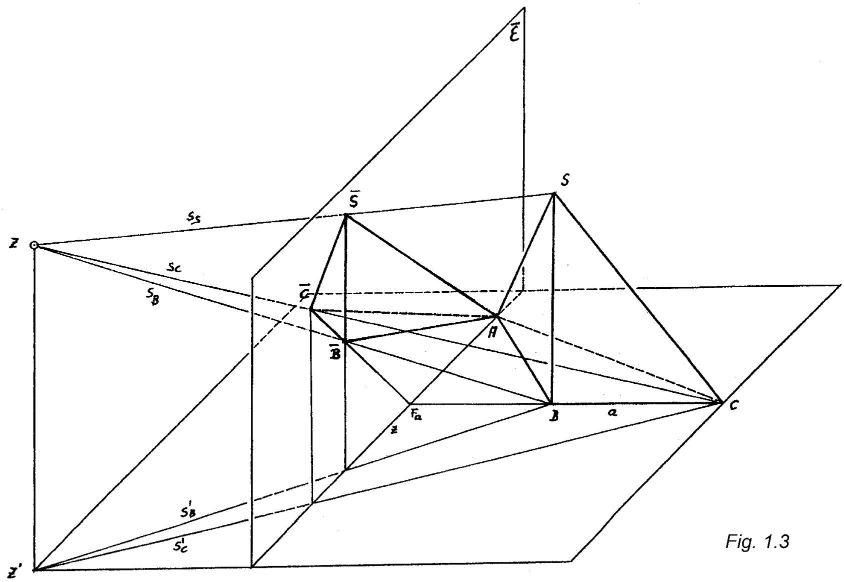


Fig. 1.3

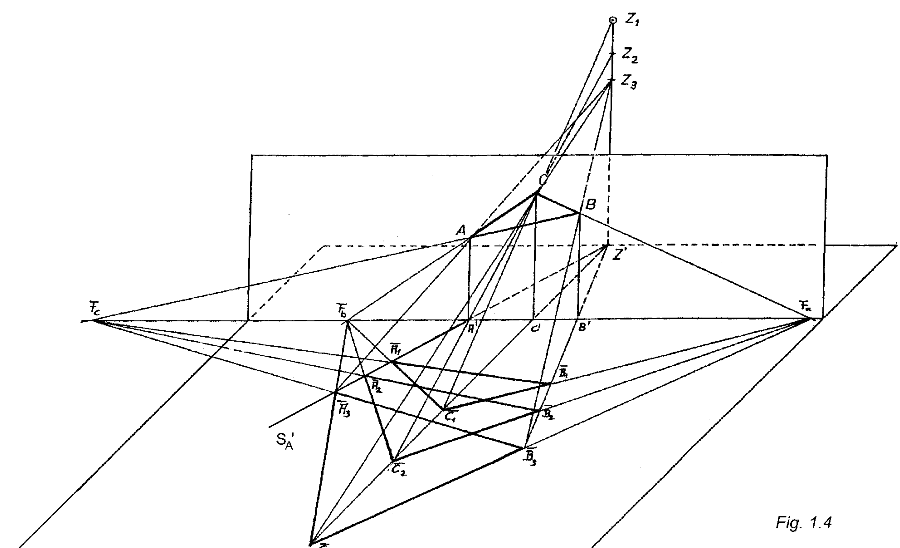


Fig. 1.4

First we'll direct our attention to the point A: As Z_1 sinks down (positions Z_2 and Z_3), the image points $\overline{A_1}$, $\overline{A_2}$, $\overline{A_3}$ move along the fixed projection line s'_A of the projection rays rotating about point A. The same applies to the vertices B and C. The triangle's sides rotate around their fixed points F_a, F_b and F_c.

We can formulate two laws:

I. The vertices of the triangle move along (fixed) straight lines, which go through a *center* Z'.

II. The sides of the triangle rotate about fixed points, which lie in a *line* (axis) z.

We call such a controlled movement a **perspective collineation** or **homology**.

A second example of a homology is:

5. Problem: Project the square $ABCD$ from the center Z_1 onto the horizontal plane and then let the center sink down to points Z_2 and Z_3 (Z_3 has a special position!).

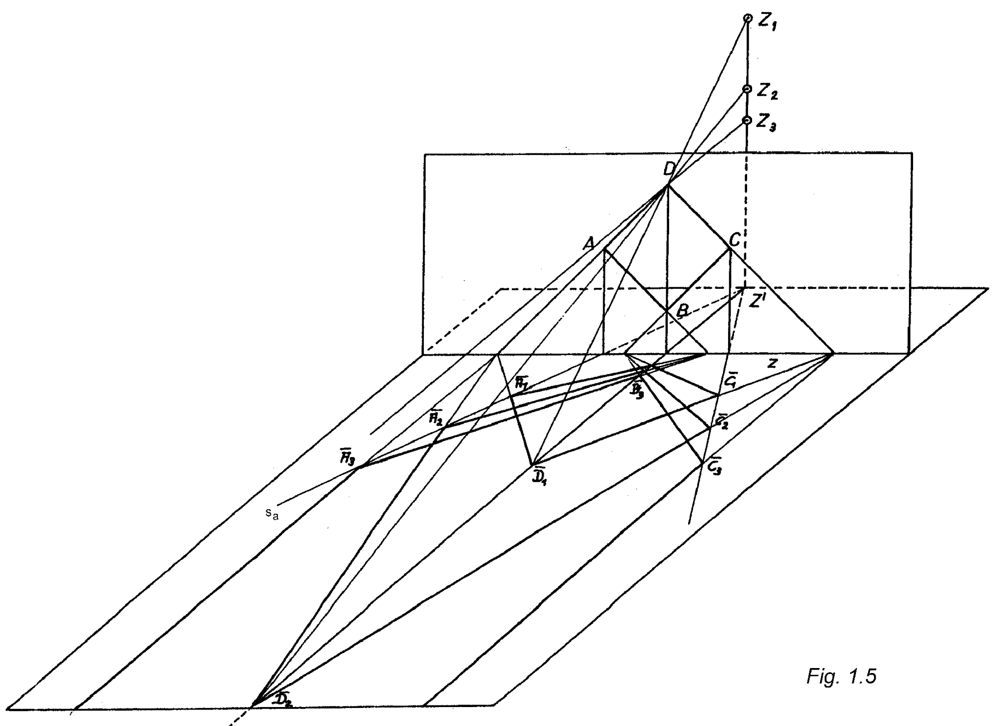


Fig. 1.5

2 The theorem of Desargues

When we project the whole configuration of Figure 1.2 onto a third plane, that is to say the original triangle ABC, the image triangle $\overline{A}\,\overline{B}\,\overline{C}$, the center Z, the axis z, the lines a, b, c and $\overline{a}, \overline{b}, \overline{c}$, the projection rays s_a, s_b, s_c, and the fixed points F_a, F_b, F_c, we get the following picture:

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