Flight Dynamics Summary

1. Introduction

In this summary we examine the flight dynamics of aircraft. But before we do that, we must examine some basic ideas necessary to explore the secrets of flight dynamics.

1.1 Basic concepts

1.1.1 Controlling an airplane

To control an aircraft, control surfaces are generally used. Examples are elevators, flaps and spoilers. When dealing with control surfaces, we can make a distinction between primary and secondary flight control surfaces. When **primary control surfaces** fail, the whole aircraft becomes uncontrollable. (Examples are elevators, ailerons and rudders.) However, when **secondary control surfaces** fail, the aircraft is just a bit harder to control. (Examples are flaps and trim tabs.)

The whole system that is necessary to control the aircraft is called the **control system**. When a control system provides direct feedback to the pilot, it is called a **reversible system**. (For example, when using a mechanical control system, the pilot feels forces on his stick.) If there is no direct feedback, then we have an **irreversible system**. (An example is a fly-by-wire system.)

1.1.2 Making assumptions

In this summary, we want to describe the flight dynamics with equations. This is, however, very difficult. To simplify it a bit, we have to make some simplifying assumptions. We assume that ...

- There is a **flat Earth**. (The Earth's curvature is zero.)
- There is a **non-rotating Earth**. (No Coriolis accelerations and such are present.)
- The aircraft has **constant mass**.
- The aircraft is a **rigid body**.
- The aircraft is **symmetric**.
- There are no **rotating masses**, like turbines. (Gyroscopic effects can be ignored.)
- There is **constant wind**. (So we ignore turbulence and gusts.)

1.2 Reference frames

1.2.1 Reference frame types

To describe the position and behavior of an aircraft, we need a **reference frame** (RF). There are several reference frames. Which one is most convenient to use depends on the circumstances. We will examine a few.

- First let's examine the **inertial reference frame** F_I. It is a right-handed orthogonal system. Its origin A is the center of the Earth. The Z_I axis points North. The X_I axis points towards the **vernal equinox**. The Y_I axis is perpendicular to both the axes. Its direction can be determined using the right-hand rule.
- In the **(normal) Earth-fixed reference frame** F_E, the origin O is at an arbitrary location on the ground. The Z_E axis points towards the ground. (It is perpendicular to it.) The X_E axis is directed North. The Y_E axis can again be determined using the right-hand rule.
- The **body-fixed reference frame** F_b is often used when dealing with aircraft. The origin of the reference frame is the center of gravity (CG) of the aircraft. The X_b axis lies in the symmetry plane of the aircraft and points forward. The Z_b axis also lies in the symmetry plane, but points downwards. (It is perpendicular to the X_b axis.) The Y_b axis can again be determined using the right-hand rule.
- The **stability reference frame** F_S is similar to the body-fixed reference frame F_b. It is rotated by an angle α_a about the Y_b axis. To find this α_a, we must examine the **relative wind vector** $\mathbf{V_a}$. We can project this vector onto the plane of symmetry of the aircraft. This projection is then the direction of the X_S axis. (The Z_S axis still lies in the plane of symmetry. Also, the Y_S axis is still equal to the Y_b axis.) So, the relative wind vector lies in the $X_S Y_S$ plane. This reference frame is particularly useful when analyzing flight dynamics.
- The **aerodynamic (air-path) reference frame** F_a is similar to the stability reference frame F_S. It is rotated by an angle β_a about the Z_S axis. This is done, such that the X_a axis points in the direction of the relative wind vector $\mathbf{V_a}$. (So the X_a axis generally does not lie in the symmetry plane anymore.) The Z_a axis is still equation to the Z_S axis. The Y_a axis can now be found using the right-hand rule.
- Finally, there is the **vehicle reference frame** F_r. Contrary to the other systems, this is a left-handed system. Its origin is a fixed point on the aircraft. The X_r axis points to the rear of the aircraft. The Y_r axis points to the left. Finally, the Z_r axis can be found using the left-hand rule. (It points upward.) This system is often used by the aircraft manufacturer, to denote the position of parts within the aircraft.

1.2.2 Changing between reference frames

We've got a lot of reference frames. It would be convenient if we could switch from one coordinate system to another. To do this, we need to rotate reference frame 1, until we wind up with reference frame 2. (We don't consider the translation of reference frames here.) When rotating reference frames, **Euler angles** ϕ come in handy. The Euler angles ϕ_x, ϕ_y and ϕ_z denote rotations about the X axis, Y axis and Z axis, respectively.

We can go from one reference frame to any other reference frame, using at most three Euler angles. An example transformation is $\phi_x \to \phi_y \to \phi_z$. In this transformation, we first rotate about the X axis, followed by a transformation about the Y axis and the Z axis, respectively. The order of these rotations is very important. Changing the order will give an entirely different final result.

1.2.3 Transformation matrices

An Euler angle can be represented by a **transformation matrix** $\mathbb{T}$. To see how this works, we consider a vector $\mathbf{x^1}$ in reference frame 1. The matrix $\mathbb{T}_{21}$ now calculates the coordinates of the same vector $\mathbf{x^2}$ in reference frame 2, according to $\mathbf{x^2} = \mathbb{T}_{21}\mathbf{x^1}$.

Let's suppose we're only rotating about the X axis. In this case, the transformation matrix $\mathbb{T}_{21}$ is quite simple. In fact, it is

$$\mathbb{T}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & \sin\phi_x \\ 0 & -\sin\phi_x & \cos\phi_x \end{bmatrix}. \tag{1.2.1}$$

Similarly, we can rotate about the Y axis and the Z axis. In this case, the transformation matrices are, respectively,

$$\mathbb{T}_{21} = \begin{bmatrix} \cos\phi_y & 0 & -\sin\phi_y \\ 0 & 1 & 0 \\ \sin\phi_y & 0 & \cos\phi_y \end{bmatrix} \qquad \text{and} \qquad \mathbb{T}_{21} = \begin{bmatrix} \cos\phi_z & \sin\phi_z & 0 \\ -\sin\phi_z & \cos\phi_z & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.2.2}$$

A sequence of rotations (like $\phi_x \to \phi_y \to \phi_z$) is now denoted by a sequence of matrix multiplications $\mathbb{T}_{41} = \mathbb{T}_{43}\mathbb{T}_{32}\mathbb{T}_{21}$. In this way, a single transformation matrix for the whole sequence can be obtained.

Transformation matrices have interesting properties. They only rotate points. They don't deform them. For this reason, the matrix columns are orthogonal. And, because the space is not stretched out either, these columns must also have length 1. A transformation matrix is thus orthogonal. This implies that

$$\mathbb{T}_{21}^{-1} = \mathbb{T}_{21}^{T} = \mathbb{T}_{12}. \tag{1.2.3}$$

1.2.4 Transformation examples

Now let's consider some actual transformations. Let's start at the body-fixed reference frame F_b. If we rotate this frame by an angle α_a about the Y axis, we find the stability reference frame F_S. If we then rotate it by an angle β_a about the Z axis, we get the aerodynamic reference frame F_a. So we can find that

$$\mathbf{x}^{\mathbf{a}} = \begin{bmatrix} \cos\beta_a & \sin\beta_a & 0 \\ -\sin\beta_a & \cos\beta_a & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}^{\mathbf{S}} = \begin{bmatrix} \cos\beta_a & \sin\beta_a & 0 \\ -\sin\beta_a & \cos\beta_a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha_a & 0 & \sin\alpha_a \\ 0 & 1 & 0 \\ -\sin\alpha_a & 0 & \cos\alpha_a \end{bmatrix} \mathbf{x}^{\mathbf{b}}. \tag{1.2.4}$$

By working things out, we can thus find that

$$\mathbb{T}_{ab} = \begin{bmatrix} \cos\beta_a\cos\alpha_a & \sin\beta_a & \cos\beta_a\sin\alpha_a \\ -\sin\beta_a\cos\alpha_a & \cos\beta_a & -\sin\beta_a\sin\alpha_a \\ -\sin\alpha_a & 0 & \cos\alpha_a \end{bmatrix}. \tag{1.2.5}$$

We can make a similar transformation between the Earth-fixed reference frame F_E and the body-fixed reference frame F_b. To do this, we first have to rotate over the **yaw angle** ψ about the Z axis. We then rotate over the **pitch angle** θ about the Y axis. Finally, we rotate over the **roll angle** φ about the X axis. If we work things out, we can find that

$$\mathbb{T}_{bE} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\varphi\sin\theta\cos\psi - \cos\varphi\sin\psi & \sin\varphi\sin\theta\sin\psi + \cos\varphi\cos\psi & \sin\varphi\cos\theta \\ \cos\varphi\sin\theta\cos\psi + \sin\varphi\sin\psi & \cos\varphi\sin\theta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\cos\theta \end{bmatrix}. \tag{1.2.6}$$

Now that's one hell of a matrix ...

1.2.5 Moving reference frames

Let's examine some point P. This point is described by vector $\mathbf{r^E}$ in reference frame F_E and by $\mathbf{r^b}$ in reference frame F_b. Also, the origin of F_b (with respect to F_E) is described by the vector $\mathbf{r_{Eb}}$. So we have $\mathbf{r^E} = \mathbf{r_{Eb}} + \mathbf{r^b}$.

Now let's examine the time derivative of $\mathbf{r^E}$ in F_E. We denote this by $\frac{d\mathbf{r^E}}{dt}\Big|_E$. It is given by

$$\frac{d\mathbf{r^E}}{dt}\bigg|_E = \frac{d\mathbf{r_{Eb}}}{dt}\bigg|_E + \frac{d\mathbf{r^b}}{dt}\bigg|_E. \tag{1.2.7}$$

Let's examine the terms in this equation. The middle term of the above equation simply indicates the movement of F_b, with respect to F_E. The right term is, however, a bit more complicated. It indicates the change of $\mathbf{r^b}$ with respect to F_E. But we usually don't know this. We only know the change of $\mathbf{r^b}$ in F_b. So we need to transform this term from F_E to F_b. Using a slightly difficult derivation, it can be shown that

$$\frac{d\mathbf{r^b}}{dt}\bigg|_E = \frac{d\mathbf{r^b}}{dt}\bigg|_b + \mathbf{\Omega_{bE}} \times \mathbf{r^b}. \tag{1.2.8}$$

The vector $\mathbf{\Omega_{bE}}$ denotes the **rotation vector** of F_b with respect to F_E. Inserting this relation into the earlier equation gives us

$$\frac{d\mathbf{r^E}}{dt}\bigg|_E = \frac{d\mathbf{r_{Eb}}}{dt}\bigg|_E + \frac{d\mathbf{r^b}}{dt}\bigg|_b + \mathbf{\Omega_{bE}} \times \mathbf{r^b}. \tag{1.2.9}$$

This is quite an important relation, so remember it well. By the way, it holds for every vector. So instead of the position vector $\mathbf{r}$, we could also take the velocity vector $\mathbf{V}$.

Finally, we note some interesting properties of the rotation vector. Given reference frames 1, 2 and 3, we have

$$\mathbf{\Omega_{12}} = -\mathbf{\Omega_{21}} \qquad \text{and} \qquad \mathbf{\Omega_{31}} = \mathbf{\Omega_{32}} + \mathbf{\Omega_{21}}. \tag{1.2.10}$$

2. Deriving the equations of motion

The flight dynamics of an aircraft are described by its **equations of motion** (EOM). We are going to derive those equations in this chapter.

2.1 Forces

2.1.1 The basic force equation

To derive the equations of motion of an aircraft, we start by examining forces. Our starting point in this is Newton's second law. However, Newton's second law only holds in an inertial reference system. Luckily, the assumptions we have made earlier imply that the Earth-fixed reference frame F_E is inertial. (However, F_b is not an inertial reference frame.) So we will derive the equations of motion with respect to F_E.

Let's examine an aircraft. Newton's second law states that

$$\mathbf{F} = \int d\mathbf{F} = \frac{d}{dt}\left(\int \mathbf{V_p}\, dm\right), \tag{2.1.1}$$

where we integrate over the entire body. It can be shown that the right part of this equation equals $\frac{d}{dt}(\mathbf{V_G} m)$, where $\mathbf{V_G}$ is the velocity of the center of gravity of the aircraft. If the aircraft has a constant mass, we can rewrite the above equation into

$$\mathbf{F} = m\frac{d\mathbf{V_G}}{dt} = m\mathbf{A_G}. \tag{2.1.2}$$

This relation looks very familiar. But it does imply something very important. The acceleration of the CG of the aircraft does not depend on how the forces are distributed along the aircraft. It only depends on the magnitude and direction of the forces.

2.1.2 Converting the force equation

There is one slight problem. The above equation holds for the F_E reference frame. But we usually work in the F_b reference frame. So we need to convert it. To do this, we can use the relation

$$\mathbf{A_G} = \left.\frac{d\mathbf{V_G}}{dt}\right|_E = \left.\frac{d\mathbf{V_G}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbf{V_G}. \tag{2.1.3}$$

Inserting this into the above equation will give

$$\mathbf{F} = m\left.\frac{d\mathbf{V_G}}{dt}\right|_b + m\,\mathbf{\Omega_{bE}} \times \mathbf{V_G} = m\begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix}. \tag{2.1.4}$$

By the way, in the above equation, we have used that

$$\mathbf{V_G} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \text{and} \qquad \mathbf{\Omega_{bE}} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{2.1.5}$$

Here, u, v and w denote the velocity components in X, Y and Z direction, respectively. Similarly, p, q and r denote rotation components about the X, Y and Z axis, respectively.

2.1.3 External forces

Let's take a look at the forces $\mathbf{F}$ our aircraft is subject to. There are two important kinds of forces: gravity and aerodynamic forces. The gravitational force $\mathbf{F_{gravity}}$ is, in fact, quite simple. It is given by

$$\mathbf{F^E_{gravity}} = \begin{bmatrix} 0 & 0 & mg \end{bmatrix}^T, \tag{2.1.6}$$

where g is the **gravitational acceleration**. The superscript E indicates that the force is given in the F_E reference frame. However, we want the force in the F_b reference frame. Luckily, we know the transformation matrix $\mathbb{T}_{bE}$. We can thus find that

$$\mathbf{F^b_{gravity}} = \mathbb{T}_{bE}\mathbf{F^E_{gravity}} = mg \begin{bmatrix} -\sin\theta \\ \sin\varphi\cos\theta \\ \cos\varphi\cos\theta \end{bmatrix}. \tag{2.1.7}$$

The aerodynamic forces $\mathbf{F_{aero}}$ are, however, a lot more difficult. For now, we won't examine them in depth. Instead, we simply say that

$$\mathbf{F^b_{aero}} = \begin{bmatrix} X^b & Y^b & Z^b \end{bmatrix}^T. \tag{2.1.8}$$

By combining this knowledge with the equation of motion for forces, we find that

$$m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} = mg \begin{bmatrix} -\sin\theta \\ \sin\varphi\cos\theta \\ \cos\varphi\cos\theta \end{bmatrix} + \begin{bmatrix} X^b \\ Y^b \\ Z^b \end{bmatrix}. \tag{2.1.9}$$

2.2 Moments

2.2.1 Angular momentum

Before we're going to look at moments, we will first examine angular momentum. The **angular momentum** of an aircraft $\mathbf{B_G}$ (with respect to the CG) is defined as

$$\mathbf{B_G} = \int d\mathbf{B_G} = \mathbf{r} \times \mathbf{V_P}\, dm, \tag{2.2.1}$$

where we integrate over every point P in the aircraft. We can substitute

$$\mathbf{V_P} = \mathbf{V_G} + \left.\frac{d\mathbf{r}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbf{r}. \tag{2.2.2}$$

If we insert this, and do a lot of working out, we can eventually find that

$$\mathbf{B_G} = \mathbb{I}_G \mathbf{\Omega_{bE}}. \tag{2.2.3}$$

The parameter $\mathbb{I}_G$ is the **inertia tensor**, with respect to the CG. It is defined as

$$\mathbb{I}_G = \begin{bmatrix} I_x & -J_{xy} & -J_{xz} \\ -J_{xy} & I_y & -J_{yz} \\ -J_{xz} & -J_{yz} & I_z \end{bmatrix} = \begin{bmatrix} \int(r_y^2 + r_z^2)dm & -\int(r_x r_y)dm & -\int(r_x r_z)dm \\ -\int(r_x r_y)dm & \int(r_x^2 + r_z^2)dm & -\int(r_y r_z)dm \\ -\int(r_x r_z)dm & -\int(r_y r_z)dm & \int(r_x^2 + r_y^2)dm \end{bmatrix}. \tag{2.2.4}$$

We have assumed that the XZ-plane of the aircraft is a plane of symmetry. For this reason, $J_{xy} = J_{yz} = 0$. This simplifies the inertia tensor a bit.

2.2.2 The moment equation

It is now time to look at moments. We again do this from the inertial reference frame F_E. The moment acting on our aircraft, with respect to its CG, is given by

$$\mathbf{M_G} = \int d\mathbf{M_G} = \int \mathbf{r} \times d\mathbf{F} = \int \mathbf{r} \times \frac{d\left(\mathbf{V_p}\, dm\right)}{dt}, \tag{2.2.5}$$

where we integrate over the entire body. Luckily, we can simplify the above relation to

$$\mathbf{M_G} = \left.\frac{d\mathbf{B_G}}{dt}\right|_E. \tag{2.2.6}$$

The above relation only holds for inertial reference frames, such as F_E. However, we want to have the above relation in F_b. So we rewrite it to

$$\mathbf{M_G} = \left.\frac{d\mathbf{B_G}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbf{B_G}. \tag{2.2.7}$$

By using $\mathbf{B_G} = \mathbb{I}_G \mathbf{\Omega_{bE}}$, we can continue to rewrite the above equation. We eventually wind up with

$$\mathbf{M_G} = \mathbb{I}_G \left.\frac{d\mathbf{\Omega_{bE}}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbb{I}_G \mathbf{\Omega_{bE}}. \tag{2.2.8}$$

In matrix-form, this equation can be written as

$$\mathbf{M_G} = \begin{bmatrix} I_x\dot{p} + (I_z - I_y)qr - J_{xz}(pq + \dot{r}) \\ I_y\dot{q} + (I_x - I_z)pr + J_{xz}(p^2 - r^2) \\ I_z\dot{r} + (I_y - I_x)pq + J_{xz}(qr - \dot{p}) \end{bmatrix}. \tag{2.2.9}$$

Note that we have used the fact that $J_{xy} = J_{yz} = 0$.

2.2.3 External moments

Let's take a closer look at $\mathbf{M_G}$. Again, we can distinguish two types of moments, acting on our aircraft. There are moments caused by gravity, and moments caused by aerodynamic forces. Luckily, the moments caused by gravity are zero. (The resultant gravitational force acts in the CG.) So we only need to consider the moments caused by aerodynamic forces. We denote those as

$$\mathbf{M^b_{G,aero}} = \begin{bmatrix} L & M & N \end{bmatrix}^T. \tag{2.2.10}$$

This turns the moment equation into

$$\begin{bmatrix} I_x\dot{p} + (I_z - I_y)qr - J_{xz}(pq + \dot{r}) \\ I_y\dot{q} + (I_x - I_z)pr + J_{xz}(p^2 - r^2) \\ I_z\dot{r} + (I_y - I_x)pq + J_{xz}(qr - \dot{p}) \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}. \tag{2.2.11}$$

2.3 Kinematic relations

2.3.1 Translational kinematics

Now that we have the force and moment equations, we only need to find the kinematic relations for our aircraft. First, we examine translational kinematics. This concerns the velocity of the CG of the aircraft with respect to the ground.

The velocity of the CG, with respect to the ground, is called the **kinematic velocity $\mathbf{V_k}$**. In the F_E reference system, it is described by

$$\mathbf{V_k} = \begin{bmatrix} V_N & V_E & -V_Z \end{bmatrix}^T. \tag{2.3.1}$$

In this equation, V_N is the velocity component in the Northward direction, V_E is the velocity component in the eastward direction, and $-V_Z$ is the vertical velocity component. (The minus sign is present because, in the Earth-fixed reference system, V_Z is defined to be positive downward.)

However, in the F_b reference system, the velocity of the CG, with respect to the ground, is given by

$$\mathbf{V_G} = \begin{bmatrix} u & v & w \end{bmatrix}^T. \tag{2.3.2}$$

To relate those two vectors to each other, we need a transformation matrix. This gives us

$$\mathbf{V_k} = \mathbb{T}_{Eb}\mathbf{V_G} = \mathbb{T}_{bE}^T\mathbf{V_G}. \tag{2.3.3}$$

This is the translational kinematic relation. We can use it to derive the change of the aircraft position. To do that, we simply have to integrate the velocities. We thus have

$$x(t) = \int_0^t V_N\,dt, \qquad y(t) = \int_0^t V_E\,dt \quad \text{and} \quad h(t) = \int_0^t -V_Z\,dt. \tag{2.3.4}$$

2.3.2 Rotational kinematics

Now let's examine rotational kinematics. This concerns the rotation of the aircraft. In the F_E reference system, the rotational velocity is described by the variables $\dot{\varphi}$, $\dot{\theta}$ and $\dot{\psi}$. However, in the F_b reference system, the rotational velocity is described by p, q and r. The relation between these two triples can be shown to be

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\varphi & \cos\theta\sin\varphi \\ 0 & -\sin\varphi & \cos\theta\cos\varphi \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{2.3.5}$$

This is the rotational kinematic relation. It is interesting to note that, if $\varphi = \theta = \psi = 0$, then $p = \dot{\varphi}$, $q = \dot{\theta}$ and $r = \dot{\psi}$. By the way, we can also invert the above relation. We would then get

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\varphi\tan\theta & \cos\varphi\tan\theta \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi/\cos\theta & \cos\varphi/\cos\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{2.3.6}$$

3. Rewriting the equations of motion

The equations of motion are quite difficult to deal with. To get some useful data out of them, we need to make them a bit simpler. For that, we first linearize them. We then simplify them. And after that, we set them in a non-dimensional form.

3.1 Linearization

3.1.1 The idea behind linearization

Let's suppose we have some non-linear function $f(\mathbf{X})$. Here, $\mathbf{X}$ is the **state** of the system. It contains several **state variables**. To linearize $f(\mathbf{X})$, we should use a multi-dimensional Taylor expansion. We then get

$$f(\mathbf{X}) \approx f(\mathbf{X_0}) + f_{X_1}(\mathbf{X_0})\Delta X_1 + f_{X_2}(\mathbf{X_0})\Delta X_2 + \ldots\ldots + f_{X_n}(\mathbf{X_0})\Delta X_n + \text{higher order terms.} \quad (3.1.1)$$

Here, $\mathbf{X_0}$ is the **initial point** about which we linearize the system. The linearization will only be valid close to this point. Also, the term ΔX_i indicates the deviation of variable X_i from the initial point $\mathbf{X_0}$.

When applying linearization, we always neglect higher order terms. This significantly simplifies the equation. (Although it's still quite big.)

3.1.2 Linearizing the states

Now let's apply linearization to the force and moment equations. We start at the right side: the states. We know from the previous chapter that

$$F_x = m(\dot{u} + qw - rv), \quad (3.1.2)$$
$$F_y = m(\dot{v} + ru - pw), \quad (3.1.3)$$
$$F_z = m(\dot{w} + pv - qu). \quad (3.1.4)$$

So we see that $F_x = f(\dot{u}, q, w, r, v)$. The state vector now consists of five states. By applying linearization, we find that

$$F_x = m(\dot{u_0} + q_0 w_0 - r_0 v_0) + m(\Delta\dot{u} + q_0\Delta w + w_0\Delta q - r_0\Delta v - v_0\Delta r), \quad (3.1.5)$$
$$F_y = m(\dot{v_0} + r_0 u_0 - p_0 w_0) + m(\Delta\dot{v} + r_0\Delta u + u_0\Delta r - p_0\Delta w - w_0\Delta p), \quad (3.1.6)$$
$$F_z = m(\dot{w_0} + p_0 v_0 - q_0 u_0) + m(\Delta\dot{w} + p_0\Delta v + v_0\Delta p - q_0\Delta u - u_0\Delta q). \quad (3.1.7)$$

We can apply a similar trick for the moments. This would, however, give us quite big expressions. And since we don't want to spoil too much paper (safe the rainforests!), we will not derive those here. Instead, we will only examine the final result in the end.

3.1.3 Linearizing the forces

Now let's try to linearize the forces. Again, we know from the previous chapter that

$$F_x = -W\sin\theta + X, \quad (3.1.8)$$
$$F_y = W\sin\psi\cos\theta + Y, \quad (3.1.9)$$
$$F_z = W\cos\psi\cos\theta + Z, \quad (3.1.10)$$

where the **weight** $W = mg$. We see that this time $F_x = f(\theta, X)$. Also, $F_y = f(\psi, \theta, Y)$ and $F_Z = f(\psi, \theta, Z)$. It may seem that linearization is easy this time. However, there are some problems.

The problems are the forces X, Y and Z. They are not part of the state of the aircraft. Instead they also depend on the state of the aircraft. And they don't only depend on the current state, but on the entire history of states! (For example, a change in angle of attack could create disturbances at the wing. These disturbances will later result in forces acting on the tail of the aircraft.)

How do we put this into equations? Well, we say that X is not only a function of the velocity u, but also of all its derivatives $\dot{u}, \ddot{u}, \ldots$. And the same goes for v, w, p, q and r. This gives us an infinitely big equation. (Great....) But luckily, experience has shown that we can neglect most of these time derivatives, as they aren't very important. There are only four exceptions. $\dot{v}$ strongly influences the variables Y and N. Also, $\dot{w}$ strongly influences Z and M. We therefore say that

$$F_x = f(\theta, X) \qquad \text{with} \qquad X = f(u, v, w, p, q, r, \delta_a, \delta_e, \delta_r, \delta_t), \tag{3.1.11}$$

$$F_y = f(\psi, \theta, Y) \qquad \text{with} \qquad Y = f(u, v, w, \dot{v}, p, q, r, \delta_a, \delta_e, \delta_r), \tag{3.1.12}$$

$$F_z = f(\psi, \theta, Z) \qquad \text{with} \qquad Z = f(u, v, w, \dot{w}, p, q, r, \delta_a, \delta_e, \delta_r, \delta_t). \tag{3.1.13}$$

When creating the Taylor expansion, we have to apply the chain rule. We then find that

$$F_x(\mathbf{X}) \approx F_x(\mathbf{X_0}) - W\cos\theta_0\Delta\theta + X_u\Delta u + X_v\Delta v + X_w\Delta w + X_p\Delta p + \ldots + X_{\delta_t}\Delta\delta_t, \tag{3.1.14}$$

$$F_y(\mathbf{X}) \approx F_y(\mathbf{X_0}) - W\sin\psi_0\sin\theta_0\Delta\theta + W\cos\psi_0\cos\theta_0\Delta\psi + Y_{\dot{v}}\Delta\dot{v} + \ldots + Y_{\delta_r}\Delta\delta_r, \tag{3.1.15}$$

$$F_z(\mathbf{X}) \approx F_z(\mathbf{X_0}) - W\cos\psi_0\sin\theta_0\Delta\theta - W\sin\psi_0\cos\theta_0\Delta\psi + Z_{\dot{w}}\Delta\dot{w} + \ldots + Z_{\delta_t}\Delta\delta_t. \tag{3.1.16}$$

Now that's one big Taylor expansion. And we haven't even written down all terms of the equation. (Note the dots in the equation.) By the way, the term X_u indicates the derivative $\partial X/\partial u$. Similarly, $X_v = \partial X/\partial v$, and so on.

You may wonder what δ_a, δ_e, δ_r and δ_t are. Those are the settings of the aileron, elevator, rudder and thrust. These settings of course influence the forces acting on the aircraft. We will examine those coefficients later in more detail. (You may also wonder, why doesn't Y depend on the thrust setting δ_t? This is because we assume that the direction of the thrust vector lies in the plane of symmetry.)

3.2 Simplification

3.2.1 Symmetry and asymmetry

Let's try to simplify that monstrocity of an equation of the previous part. To do that, we have to apply several tricks. The most important one, is that of symmetry and asymmetry.

We can make a distinction between symmetric and asymmetric forces/deviations. The symmetric deviations (the deviations which don't break the symmetry) are u, w and q. The symmetric forces/moments are X, Z and M. Similarly, the asymmetric deviations are v, p and r. The asymmetric forces/moments are Y, L and N.

It can now be shown that there is no coupling between the symmetric and the asymmetric properties. (That is, as long as the deviations are small.) In other words, X is uneffected by v, p and r. Thus $X_v = X_p = X_r = 0$. The same trick works for the other forces and moments as well. This causes a lot of terms to disappear in the force equations.

3.2.2 Simplifying the force equations

There is also another important trick we use, when simplifying the force equations. We assume that the aircraft is flying in a steady symmetric flight. This means that

$$\begin{array}{cccccccc} u_0 \neq 0 & \dot{u}_0 = 0 & p_0 = 0 & \dot{p}_0 = 0 & \varphi_0 = 0 & \dot{\varphi}_0 = 0 & X_0 \neq 0 & \dot{X}_0 = 0, \\ v_0 = 0 & \dot{v}_0 = 0 & q_0 = 0 & \dot{q}_0 = 0 & \theta_0 \neq 0 & \dot{\theta}_0 = 0 & Y_0 = 0 & \dot{Y}_0 = 0, \\ w_0 \neq 0 & \dot{w}_0 = 0 & r_0 = 0 & \dot{r}_0 = 0 & \psi_0 \neq 0 & \dot{\psi}_0 = 0 & Z_0 \neq 0 & \dot{Z}_0 = 0. \end{array} \tag{3.2.1}$$

This greatly simplifies the $F_x(\mathbf{X_0})$, $F_y(\mathbf{X_0})$ and $F_z(\mathbf{X_0})$ terms.

Now it is finally time to apply all these simplifications and tricks. It will give us the force equations for small deviations from a steady symmetric flight. These equations are

$$-W\cos\theta_0\theta + X_u u + X_w w + X_q q + X_{\delta_e}\delta_e + X_{\delta_t}\delta_t = m(\dot{u} + w_0 q), \tag{3.2.2}$$

$$W\cos\theta_0\psi + Y_v v + Y_{\dot{v}}\dot{v} + Y_p p + Y_r r + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r = m(\dot{v} + u_0 r - w_0 p), \tag{3.2.3}$$

$$-W\sin\theta_0\theta + Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q + Z_{\delta_e}\delta_e + Z_{\delta_t}\delta_t = m(\dot{w} - u_0 q). \tag{3.2.4}$$

Of these three equations, the first and the third correspond to symmetric motion. The second equation corresponds to asymmetric motion.

You may wonder, where did all the Δ's go to? Well, to simplify our notation, we omitted them. So in the above equation, all variables indicate the displacement from the initial position $\mathbf{X_0}$.

Finally, there is one more small simplification we could do. We haven't fully defined our reference system yet. (We haven't specified where the X axis is in the symmetry plane.) Now let's choose our reference system. The most convenient choice is in this case the stability reference frame F_S. By choosing this frame, we have $u_0 = V$ and $w_0 = 0$. (V is the velocity.) This eliminates one more term.

3.2.3 The moment equations

In a similar way, we can linearize and simplify the moment equations. We won't go through that tedious process. By now you should more or less know how linearization is done. We'll just mention the results. They are

$$L_v v + L_p p + L_r r + L_{\delta_a}\delta_a + L_{\delta_r}\delta_r = I_x\dot{p} - J_{xz}\dot{r}, \tag{3.2.5}$$

$$M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q + M_{\delta_e}\delta_e + M_{\delta_t}\delta_t = I_y\dot{q}, \tag{3.2.6}$$

$$N_v v + N_{\dot{v}}\dot{v} + N_p p + N_r r + N_{\delta_a}\delta_a + N_{\delta_r}\delta_r = I_z\dot{r} - J_{xz}\dot{p}. \tag{3.2.7}$$

Of these three equations, only the second one corresponds to symmetric motion. The other two correspond to asymmetric motion.

3.2.4 The kinematic relations

The kinematic relations can also be linearized. (This is, in fact, not that difficult.) After we apply the simplifications, we wind up with

$$\dot{\varphi} = p + r\tan\theta_0, \tag{3.2.8}$$

$$\dot{\theta} = q, \tag{3.2.9}$$

$$\dot{\psi} = \frac{r}{\cos\theta_0}. \tag{3.2.10}$$

Of these three equations, only the second one corresponds to symmetric motion. The other two correspond to asymmetric motion.

3.3 Setting the equations in a non-dimensional form

3.3.1 The dividing term

Aerospace engineers often like to work with non-dimensional coefficients. By doing this, they can easily compare aircraft of different size and weight. So, we will also try to make our equations non-dimensional. But how do we do that? We simply divide the equations by a certain value, making them non-dimensional.

The question remains, by what do we divide them? Well, we divide the force equations by $\frac{1}{2}\rho V^2 S$, the symmetric moment equation by $\frac{1}{2}\rho V^2 S\bar{c}$, the asymmetric moment equations by $\frac{1}{2}\rho V^2 Sb$, the symmetric kinematic equation by $V/\bar{c}$ and the asymmetric kinematic equations by V/b. Here, S is the wing surface area, $\bar{c}$ is the mean chord length, and b is the wing span. (Note that we use $\bar{c}$ for symmetric equations, while we use b for asymmetric equations.)

3.3.2 Defining coefficients

Dividing our equations by a big term won't make them look prettier. To make them still readable, we need to define some coefficients. To see how we do that, we consider the term $X_u u$. We have divided this term by $\frac{1}{2}\rho V^2 S$. We can now rewrite this term to

$$\frac{X_u u}{\frac{1}{2}\rho V^2 S} = \frac{X_u}{\frac{1}{2}\rho V S}\frac{u}{V} = C_{X_u}\hat{u}. \tag{3.3.1}$$

In this equation, we have defined the **non-dimensional velocity** $\hat{u}$. There is also the coefficient $C_{X_u} = X_u/(\frac{1}{2}\rho V S)$. This coefficient is called a **stability derivative**.

We can apply the same trick to other terms as well. For example, we can rewrite the term $X_w w$ to

$$\frac{X_w w}{\frac{1}{2}\rho V^2 S} = \frac{X_w}{\frac{1}{2}\rho V S}\frac{w}{V} = C_{X_\alpha}\alpha, \tag{3.3.2}$$

where the **angle of attack** α is approximated by $\alpha = w/V$. We can also rewrite the term $X_q q$ to

$$\frac{X_q q}{\frac{1}{2}\rho V^2 S} = \frac{X_q}{\frac{1}{2}\rho V^2 S\bar{c}}\frac{\bar{c}}{V}\frac{d\theta}{dt} = C_{X_q} D_c\theta. \tag{3.3.3}$$

This time we don't only see a new coefficient. There is also the **differential operator** D_c. Another differential operator is D_b. D_c and D_b are defined as

$$D_c = \frac{\bar{c}}{V}\frac{d}{dt} \qquad \text{and} \qquad D_b = \frac{b}{V}\frac{d}{dt}. \tag{3.3.4}$$

In this way, a lot of coefficients can be defined. We won't state the definitions of all the coefficients here. (There are simply too many for a summary.) But you probably can guess the meaning of most of them by now. And you simply have to look up the others.

3.3.3 The equations of motion in matrix form

So, we could now write down a new set of equations, with a lot of coefficients. However, we know that these equations are linear. So, we can put them in a matrix form. If we do that, we will find two interesting matrix equations. The equations for the symmetric motion are given by

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) D_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{X_{\delta_e}} & -C_{X_{\delta_t}} \\ -C_{Z_{\delta_e}} & -C_{Z_{\delta_t}} \\ 0 & 0 \\ -C_{m_{\delta_e}} & -C_{m_{\delta_t}} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}. \tag{3.3.5}$$

You may note that, instead of using the subscript M, we use the subscript m. This is just a writing convention. You also haven't seen the variable K_Y^2 yet. It is defined as $K_Y^2 = \frac{I_y}{m\bar{c}^2}$. Also, $\mu_c = \frac{m}{\rho S\bar{c}}$.

The equations for the asymmetric motion are given by

$$\begin{bmatrix} C_{Y_\beta} + (C_{Y_{\dot\beta}} - 2\mu_b)D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} + C_{n_{\dot\beta}} D_b & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \begin{bmatrix} -C_{Y_{\delta_a}} & -C_{Y_{\delta_r}} \\ 0 & 0 \\ -C_{l_{\delta_a}} & -C_{l_{\delta_r}} \\ -C_{n_{\delta_a}} & -C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}. \tag{3.3.6}$$

Again, note that, instead of using the subscripts L and N, we have used l and n. Also, the **slip angle** β is defined as $\beta = v/V$.

3.3.4 Equations of motion in state-space form

We can also put our equation in state-space form, being

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \qquad \text{and} \qquad \mathbf{y} = C\mathbf{x} + D\mathbf{u}. \tag{3.3.7}$$

Here, A is the **state matrix**, B is the **input matrix**, C is the **output matrix** and D is the **direct matrix**. Since the system is time-invariant, all these matrices are constant. Also, $\mathbf{x}$ is the **state vector**, $\mathbf{u}$ is the **input vector** and $\mathbf{y}$ is the **output vector**.

The state-space form has several advantages. First of all, the parameters can be solved for at every time t. (The complicated equations for this are known.) Second, computers are very good at performing simulations, once a situation has been described in state-space form.

After some interesting matrix manipulation, the state-space form of the symmetric motions can be derived. The result is

$$\begin{bmatrix} \dot{\hat{u}} \\ \dot\alpha \\ \dot\theta \\ \frac{\dot{q}\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} x_u & x_\alpha & x_\theta & 0 \\ z_u & z_\alpha & z_\theta & z_q \\ 0 & 0 & 0 & \frac{V}{\bar{c}} \\ m_u & m_\alpha & m_\theta & m_q \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} + \begin{bmatrix} x_{\delta_e} & x_{\delta_t} \\ z_{\delta_e} & z_{\delta_t} \\ 0 & 0 \\ m_{\delta_e} & m_{\delta_t} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}. \tag{3.3.8}$$

There are quite some strange new coefficients in this equation. The equations, with which these coefficients are calculated, can be looked up. However, we will not mention those here.

You may notice that, in the above equation, we only have the state matrix A and the input matrix B. The matrices C and D are not present. That is because they depend on what output you want to get out of your system. So we can't generally give them here. They are often quite trivial though.

Similarly, the state-space form of the asymmetric motions can be found. This time we have

$$\begin{bmatrix} \dot\beta \\ \dot\varphi \\ \frac{\dot{p}b}{2V} \\ \frac{\dot{r}b}{2V} \end{bmatrix} = \begin{bmatrix} y_\beta & y_\varphi & y_p & y_r \\ 0 & 0 & \frac{2V}{b} & 0 \\ l_\beta & 0 & l_p & l_r \\ n_\beta & 0 & n_p & n_r \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} + \begin{bmatrix} 0 & y_{\delta_r} \\ 0 & 0 \\ l_{\delta_a} & l_{\delta_r} \\ n_{\delta_a} & n_{\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}. \tag{3.3.9}$$

And that concludes this collection of oversized equations.

4. The aerodynamic center

In this chapter, we're going to focus on the aerodynamic center, and its effect on the moment coefficient C_m.

4.1 Force and moment coefficients

4.1.1 Aerodynamic forces

Let's investigate a wing. This wing is subject to a pressure distribution. We can sum up this entire pressure distribution. This gives us a resultant **aerodynamic force vector $\mathbf{C_R}$**.

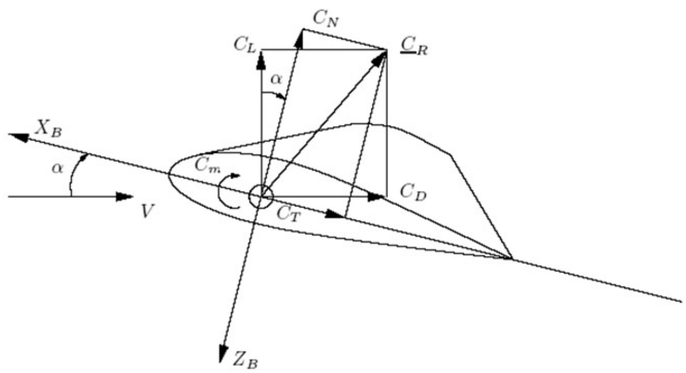

Figure 4.1: The forces and moments acting on a wing.

Let's split up the aerodynamic force vector $\mathbf{C_R}$. We can do this in multiple ways. We can split the force up into a (dimensionless) **normal force coefficient** C_N and a **tangential force coefficient** C_T. We can also split it up into a **lift force coefficient** C_L and a **drag force coefficient** C_D. Both methods are displayed in figure 4.1. The relation between the four force coefficients is given by

$$C_N = C_L \cos\alpha + C_D \sin\alpha, \tag{4.1.1}$$
$$C_T = C_D \cos\alpha - C_L \sin\alpha. \tag{4.1.2}$$

The coefficients C_L, C_N, C_T and C_D all vary with the angle of attack α. So if the angle of attack changes, so do those coefficients.

4.1.2 The aerodynamic moment

Next to the aerodynamic forces, we also have an **aerodynamic moment coefficient** C_m. This moment depends on the reference position. Let's suppose we know the moment coefficient $C_{m_{(x_1,z_1)}}$ about some point (x_1, z_1). The moment coefficient $C_{m_{(x_2,z_2)}}$ about some other point (x_2, z_2) can now be found using

$$C_{m_{(x_2,z_2)}} = C_{m_{(x_1,z_1)}} + C_N \frac{x_2 - x_1}{\bar{c}} - C_T \frac{z_2 - z_1}{\bar{c}}. \tag{4.1.3}$$

Here, $\bar{c}$ is the **mean aerodynamic chord length**. (The **mean aerodynamic chord** (MAC) can be seen as the 'average' chord of the entire 3D wing.) Also, x and z denote the position of the reference point in the vehicle reference frame F_r. We define (x_0, z_0) to be the position of the leading edge of the MAC.

4.2 Important points of the wing

4.2.1 The center of pressure

Let's put our reference point (x, z) (for calculating C_m) on the chord. (We thus set $z = z_0$.) There now is a certain value of x, for which $C_m = 0$. This point is called the **center of pressure** (CP). We denote its coordinates by (x_d, z_0). (The CP is the point where the line of action of $\mathbf{C_R}$ crosses the chord.)

Let's suppose that we know the moment coefficient $C_{m_{(x_0,z_0)}}$ about the leading edge. We can then find x_d using

$$C_{m_{(x_d,z_0)}} = 0 = C_{m_{(x_0,z_0)}} + C_N \frac{x_d - x_0}{\bar{c}}. \tag{4.2.1}$$

Let's define $e = x_d - x_0$. We can then find that

$$\frac{e}{\bar{c}} = -\frac{C_{m_{(x_0,z_0)}}}{C_N}. \tag{4.2.2}$$

4.2.2 Lines and metacenters

Let's examine a wing at a certain angle of attack α. This wing is subjected to a resultant force $\mathbf{C_R}$. For all points on the line of action of $\mathbf{C_R}$, we have $C_m = 0$.

Now let's examine all points for which $dC_m/d\alpha = 0$. These points also lie on one line. This line is called the **neutral line**. The point where this line crosses the MAC (and thus $z = z_0$) is called the **neutral point**. The crossing point of the neutral line and the line of action of $\mathbf{C_R}$ is called the **first metacenter** M_1. This point has both $C_m = 0$ and $dC_m/d\alpha = 0$.

Let's take another look at the neutral line. On this line is a point for which $d^2C_m/d\alpha^2 = 0$. This point is called the **second metacenter**.

It is important to remember that all the lines and points discussed above change as α changes. However, the second metacenter changes only very little. We therefore assume that its position is constant for different angles of attack α.

4.2.3 The aerodynamic center

Previously, we have defined the second metacenter. However, in aerodynamics, we usually refer to this point as the **aerodynamic center** (AC). Its coordinates are denoted by (x_{ac}, z_{ac}). The corresponding moment coefficient is written as $C_{m_{ac}}$. We know that we have $dC_{m_{ac}}/d\alpha = 0$ and $d^2C_{m_{ac}}/d\alpha^2 = 0$. We can use this to find x_{ac} and z_{ac}.

To find x_{ac} and z_{ac}, we have to differentiate equation (4.1.3) with respect to α. Differentiating it once gives

$$\frac{dC_{m_{ac}}}{d\alpha} = 0 = \frac{dC_{m_{(x_0,z_0)}}}{d\alpha} + \frac{dC_N}{d\alpha}\frac{x_{ac} - x_0}{\bar{c}} - \frac{dC_T}{d\alpha}\frac{z_{ac} - z_0}{\bar{c}}. \tag{4.2.3}$$

(Note that we have used the fact that the position of the AC doesn't vary with α.) Differentiating it twice gives

$$\frac{d^2C_{m_{ac}}}{d\alpha^2} = 0 = \frac{d^2C_{m_{(x_0,z_0)}}}{d\alpha^2} + \frac{d^2C_N}{d\alpha^2}\frac{x_{ac} - x_0}{\bar{c}} - \frac{d^2C_T}{d\alpha^2}\frac{z_{ac} - z_0}{\bar{c}}. \tag{4.2.4}$$

We now have two equations and two unknowns. We can thus solve for x_{ac} and z_{ac}. After this, it is easy to find the corresponding moment coefficient $C_{m_{ac}}$. And since $dC_{m_{ac}}/d\alpha = 0$, we know that this moment coefficient stays the same, even if α varies.

We have just described an analytical method to find the AC. There are also graphical methods to find the AC. We won't go into detail on those methods though.

4.2.4 Simplifications

We can make a couple of simplifications. Usually, $dC_T/d\alpha$ is rather small compared to $dC_N/d\alpha$. We therefore often neglect the effects of the tangential force coefficient C_T. If we do this, we find that the AC lies on the MAC ($z_{ac} = z_0$). In fact, the AC coincides with the neutral point.

Finding the position of the AC has now become a lot easier. We know that $z_{ac} = z_0$. We can use this to show that x_{ac} satisfies

$$\frac{x_{ac} - x_0}{\bar{c}} = -\frac{dC_{m_{(x_0,z_0)}}}{dC_N}. \tag{4.2.5}$$

Once x_{ac} has been determined, we can find the moment coefficient about any other point on the wing. Based on our simplifications, we have

$$C_m(x) = C_{m_{ac}} + C_N \frac{x - x_{ac}}{\bar{c}}. \tag{4.2.6}$$

We can also see another interesting fact from this equation. If $C_N = 0$, the moment coefficient is constant along the wing. And the value of this moment coefficient is equal to $C_{m_{ac}}$. In other words, the value of $C_{m_{ac}}$ is the value of C_m when $C_N = 0$. (This rule holds for every reference point.)

4.3 Static stability

4.3.1 Stability types

Let's suppose that the aircraft is performing a steady flight. The aircraft is then in equilibrium. This means that the moment coefficient about the center of gravity (CG) must be 0. ($C_{m_{cg}} = 0$.) Now let's suppose that the aircraft gets a small deviation from this steady flight. For example, α increases to $\alpha + d\alpha$. What happens?

Due to the change in angle of attack, $C_{m_{cg}}$ is no longer zero. Instead, it will get a value of $dC_{m_{cg}}$. We can now distinguish three cases.

- The change in moment $dC_{m_{cg}}$ is in the same direction as $d\alpha$. We thus have $dC_{m_{cg}}/d\alpha > 0$. In this case, the moment causes α to diverge away from the equilibrium position. The aircraft is therefore **unstable**.
- The change in moment $dC_{m_{cg}}$ is directed oppositely to $d\alpha$. We now have $dC_{m_{cg}}/d\alpha < 0$. In this case, the moment causes α to get back to its equilibrium position. The aircraft is thus **stable**.
- The change in moment $dC_{m_{cg}} = 0$, and thus also $dC_{m_{cg}}/d\alpha = 0$. In this case, we are in a new equilibrium position. This situation is called **neutrally stable** or **indifferent**.

4.3.2 The position of the center of gravity

We just saw that, to have a stable aircraft, we should have $dC_{m_{cg}}/d\alpha < 0$. It turns out that the position of the CG is very important for this. To see why, we differentiate equation (4.2.6) with respect to α. We find that

$$\frac{dC_{m_{cg}}}{d\alpha} = \frac{dC_N}{d\alpha} \frac{x_{cg} - x_{ac}}{\bar{c}}. \tag{4.3.1}$$

In general, $dC_N/d\alpha > 0$. So, to have a stable aircraft, we must have $x_{cg} - x_{ac} < 0$. The aerodynamic center should thus be more to the rear of the aircraft than the CG. (This is also why airplanes have a stabilizing horizontal tailplane: It moves the aerodynamic center to the rear.)

4.4 Three-dimensional wings

4.4.1 Basic and additional lift distributions

Previously, we have only examined 2D wings. We will now examine a 3D wing. The wing has a wing span b. Also, at every point y, the 2D airfoil has its own chord $c(y)$ and lift coefficient $c_l(y,\alpha)$. It also has a contribution to the lift. By summing up all these lift contributions, we can find the total lift coefficient C_L of the wing. This goes according to

$$C_L \frac{1}{2}\rho V^2 S = 2\int_0^{b/2} c_l(y)\frac{1}{2}\rho V^2 c(y)\,dy \qquad \Rightarrow \qquad C_L = 2\int_0^{b/2} c_l(y)\frac{c(y)}{\bar{c}}dy. \tag{4.4.1}$$

Note that we have used that $S = b\bar{c}$. We also have used the assumption that the wing is symmetric, by integrating over only one half of the wing.

We can split the lift coefficient distribution $c_l(y,\alpha)$ up into two parts. First, there is the **basic lift distribution** $c_{l_b}(y)$. This is the lift distribution corresponding to the **zero-lift angle of attack** $\alpha_{C_L=0}$. (So $c_{l_b}(y) = c_l(y,\alpha_{C_L=0})$.) Per definition, we thus have

$$2\int_0^{b/2} c_{l_b}(y)\,\frac{c(y)}{\bar{c}}\,dy = 0. \tag{4.4.2}$$

Second, there is the **additional lift distribution** $c_{l_a}(y,\alpha)$. This lift distribution takes into account changes in α. It is defined as $c_{l_a}(y,\alpha) = c_l(y,\alpha) - c_{l_b}(y)$. So, if we have $\alpha = \alpha_{C_L=0}$, then $c_{l_a}(y,\alpha_{C_L=0}) = 0$ for all y.

4.4.2 The aerodynamic center of a 3D wing

You may wonder, what is the use of splitting up the lift distribution? Well, it can be shown that the position of the aerodynamic center of the entire wing $\bar{x}_{ac}$ only depends on c_{l_a}. In fact, we have

$$\frac{\bar{x}_{ac} - \bar{x}_0}{\bar{c}} = \frac{1}{C_L}\frac{2}{S\bar{c}}\int_0^{b/2} c_{l_a}(y,\alpha)\,c(y)\,(x_{ac}(y) - \bar{x}_0)\,dy. \tag{4.4.3}$$

It is important to note the difference between all the x's. x_{ac} is the position of the AC of the 2D airfoil. $\bar{x}_{ac}$ is the position of the AC of the entire 3D wing. Finally, $\bar{x}_0$ is the position of the leading edge of the MAC. By the way, the above equation only holds for reasonable taper and wing twist angles. For very tapered/twisted wings, the above equation loses its accuracy.

Now let's examine the moment coefficient of the entire wing. This moment coefficient only depends on the moment coefficients $c_{m_{ac}}$ and the basic lift distribution c_{l_b} of the individual airfoils. In fact, it can be shown that

$$C_{m_{ac}} = \frac{2}{S\bar{c}}\left(\int_0^{b/2} c_{m_{ac}}(y)\,c(y)^2\,dy - \int_0^{b/2} c_{l_b}(y)\,c(y)\,(x_{ac}(y) - \bar{x}_0)\,dy\right). \tag{4.4.4}$$

4.4.3 Effects of the 3D wing shape

Let's investigate how the wing shape effects $\bar{x}_{ac}$ and $C_{m_{ac}}$. There are several properties that we can give to our 3D wing.

- A cambered airfoil. **Camber** causes the value $c_{m_{ac}}$ of the individual airfoils to become more negative. So $C_{m_{ac}}$ also becomes more negative. $\bar{x}_{ac}$ doesn't really change.

- A swept wing. When dealing with swept wings, the term $(x_{ac}(y) - \bar{x}_0)$ becomes important. Wings with high **sweep angles** Λ tend to have a shifting AC at high angles of attack. Whether this improves the stability or not depends on other parameters as well.
- A tapered wing. The **taper ratio** $\lambda = c_t/c_r$ (the ratio of the tip chord and the root chord) slightly influences stability. For swept back wings, a low taper ratio tends to have a stabilizing influence.
- A slender wing. The **aspect ratio** A has only little influence on the position of the AC. However, a slender wing (high A) with a large sweep angle Λ will become unstable at large angles of attack.
- A twisted wing. Applying a **wing twist angle** ε causes the basic lift distribution c_{l_b} to change. This causes $C_{m_{ac}}$ to change as well. In what way $C_{m_{ac}}$ changes, depends on the direction of the wing twist.

Predicting the exact behaviour of the wing is, however, rather difficult. A lot of parameters influence the wing behaviour. So don't be surprised if the above rules don't always hold.

5. Examining an entire aircraft

In the previous chapter, we've only considered the wing of an aircraft. Now we're going to add the rest of the aircraft too. How do the various components of the aircraft influence each other?

5.1 Adding a fuselage

5.1.1 Changes in moment coefficient

Previously, we have only considered a wing. This wing had a moment coefficient C_{m_w}. Now let's add a fuselage. The combination of wing and fuselage has a moment coefficient $C_{m_{wf}}$. The change in moment coefficient ΔC_m is now defined such that

$$C_{m_{wf}} = C_{m_w} + \Delta C_m. \tag{5.1.1}$$

Let's take a closer look at this change ΔC_m. What does it consist of? We know that a fuselage in a flow usually has a moment coefficient C_{m_f}. However, the wing causes the flow around the fuselage to change. This also causes a moment coefficient induced on the fuselage, denoted by $\Delta C_{m_{fi}}$. Finally, the fuselage effects the flow around the wing. There is thus also a factor $\Delta C_{m_{wi}}$. We thus have

$$\Delta C_m = C_{m_f} + \Delta C_{m_{fi}} + \Delta C_{m_{wi}}. \tag{5.1.2}$$

In this equation, the coefficients C_{m_f} and $\Delta C_{m_{fi}}$ are usually considered together as $C_{m_{fi}}$.

5.1.2 Effects of the fuselage

We can use inviscid incompressible flow theory to examine the fuselage. We then find that the moment coefficient of the fuselage, in the induced velocity field, is

$$C_{m_{fi}} = C_{m_f} + \Delta C_{m_{fi}} = \frac{\pi}{2S\bar{c}} \int_0^{l_f} b_f(x)^2 \, \alpha_f(x) \, dx. \tag{5.1.3}$$

Here, $b_f(x)$ is the **fuselage width** and $\alpha_f(x)$ is the (effective) **fuselage angle of attack**. We also integrate over the entire length l_f of the fuselage.

If there was only a fuselage (and no wing), then the fuselage would have a constant angle of attack. However, the wing causes the angle of attack to vary. In front of the wing, the flow goes up a bit. Behind the wing, there is a downwash. To deal with these complicated effects, we apply linearization. We thus approximate $\alpha_f(x)$ as

$$\alpha_f(x) = \alpha_{f_0} + \frac{d\alpha_f(x)}{d\alpha}(\alpha - \alpha_0). \tag{5.1.4}$$

Here, α_0 is the **zero normal force angle of attack** $\alpha_{C_N=0}$. α_{f_0} is the corresponding fuselage angle of attack. By using the above equation, we can find a relation for $C_{m_{fi}}$. We get

$$C_{m_{fi}} = \frac{\pi \alpha_{f_0}}{2S\bar{c}} \int_0^{l_f} b_f(x)^2 \, dx + \frac{\pi(\alpha - \alpha_0)}{2S\bar{c}} \int_0^{l_f} b_f(x)^2 \, \frac{d\alpha_f(x)}{d\alpha} \, dx. \tag{5.1.5}$$

5.1.3 The shift of the aerodynamic center

Adding the fuselage causes the aerodynamic center to shift. We know that

$$C_{m_w} = C_{m_{ac_w}} + C_{N_w} \frac{x - x_{ac_w}}{\bar{c}} \qquad \text{and} \qquad C_{m_{wf}} = C_{m_{ac_{wf}}} + C_{N_{wf}} \frac{x - x_{ac_{wf}}}{\bar{c}}. \tag{5.1.6}$$

Let's assume that adding the fuselage doesn't effect the normal force. Thus $C_{N_w} = C_{N_{wf}} = C_N$. In this case, we have

$$\Delta C_m = C_{m_{wf}} - C_{m_w} = \Delta C_{m_{ac}} - C_N \frac{x_{ac_{wf}} - x_{ac_w}}{\bar{c}}. \tag{5.1.7}$$

We can differentiate this equation with respect to α. From the definition of the AC follows that $d(\Delta C_{m_{ac}})/d\alpha = 0$. If we then also use the fact that $dC_N/d\alpha = C_{N_\alpha}$, we find that

$$\frac{x_{ac_{wf}} - x_{ac_w}}{\bar{c}} = \frac{\Delta x_{ac}}{\bar{c}} = -\frac{1}{C_{N_\alpha}} \frac{d(\Delta C_m)}{d\alpha}. \tag{5.1.8}$$

Part of this shift is caused by the fuselage, while the other part is caused by the new flow on the wing. The shift in angle of attack, due to the fuselage, is

$$\left(\frac{\Delta x_{ac}}{\bar{c}}\right)_{fi} = -\frac{1}{C_{N_\alpha}} \frac{dC_{m_{fi}}}{d\alpha} = -\frac{1}{C_{N_\alpha}} \frac{\pi}{2S\bar{c}} \int_0^{l_f} b_f(x)^2 \frac{d\alpha_f(x)}{d\alpha} \, dx. \tag{5.1.9}$$

The shift due to the flow induced on the wing is denoted by $\left(\frac{\Delta x_{ac}}{\bar{c}}\right)_{wi}$. We don't have a clear equation for this part of the shift. However, it is important to remember that this shift is only significant for swept wings. If there is a positive sweep angle, then the AC moves backward.

5.2 Adding the rest of the aircraft

5.2.1 The three parts

It is now time to examine an entire aircraft. The CG of this aircraft is positioned at (x_{cg}, z_{cg}). We split this aircraft up into three parts.

- First, there is the wing, with attached fuselage and nacelles. The position of the AC of this part is (x_w, z_w). Two forces and one moment are acting in this AC. There are a normal force N_w (directed upward), a tangential force T_w (directed to the rear) and a moment M_{ac_w}.
- Second, we have a horizontal tailplane. The AC of this part is at (x_h, z_h). In it are acting a normal force N_h, a tangential force T_h and a moment M_{ac_h}.
- Third, there is the propulsion unit. Contrary to the other two parts, this part has no moment. It does have a normal force N_p and a tangential force T_p. However, these forces are all tilted upward by the **thrust inclination** i_p. (So they have different directions then the force T_w, T_h, N_w and N_h.) Also, the tangential force T_p is defined to be positive when directed forward. (This is contrary to the forces T_h and T_w, which are positive when directed backward.)

5.2.2 The equations of motion

We will now derive the equations of motion for this simplified aircraft. We assume that the aircraft is in a fully symmetric flight. We then only need to consider three equations of motion. Taking the sum of forces in X direction gives

$$T = T_w + T_h - T_p \cos i_p + N_p \sin i_p = -W \sin\theta. \tag{5.2.1}$$

Similarly, we can take the sum of forces in Z direction, and the sum of moments about the CG. (This is done about the Y axis.) We then get

$$N = N_w + N_h + N_p \cos i_p + T_p \sin i_p = W \cos\theta, \tag{5.2.2}$$

$$M = M_{ac_w} + N_w(x_{cg} - x_w) - T_w(z_{cg} - z_w) + M_{ac_h} + N_h(x_{cg} - x_h) - T_h(z_{cg} - z_h) + \dots$$
$$\dots + (N_p \cos i_p + T_p \sin i_p)(x_{cg} - x_p) + (T_p \cos i_p - N_p \sin i_p)(z_{cg} - z_p) = 0. \quad (5.2.3)$$

We can simplify these equations, by making a couple of assumptions. We want to examine the stability of the aircraft. The propulsion doesn't influence the stability of the aircraft much. So we neglect propulsion effects. We also neglect T_h, since it is very small compared to T_w. We assume that $(z_{cg} - z_w) \approx 0$. And finally, we neglect M_{ac_h}. This gives us

$$T = T_w = -W \sin\theta, \quad (5.2.4)$$

$$N = N_w + N_h = W \cos\theta, \quad (5.2.5)$$

$$M = M_{ac_w} + N_w(x_{cg} - x_w) + N_h(x_{cg} - x_h) = 0. \quad (5.2.6)$$

That simplifies matters greatly.

5.2.3 Non-dimensionalizing the equations of motion

Let's non-dimensionalize the equations of motion of the previous paragraph. For that, we divide the force equations by $\frac{1}{2}\rho V^2 S$ and the moment equation by $\frac{1}{2}\rho V^2 S\bar{c}$. This then gives us

$$C_T = C_{T_w} = -\frac{W}{\frac{1}{2}\rho V^2 S} \sin\theta, \quad (5.2.7)$$

$$C_N = C_{N_w} + C_{N_h} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} = \frac{W}{\frac{1}{2}\rho V^2 S} \cos\theta, \quad (5.2.8)$$

$$C_m = C_{m_{ac_w}} + C_{N_w} \frac{x_{cg} - x_w}{\bar{c}} - C_{N_h} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} = 0. \quad (5.2.9)$$

A lot of new coefficients have suddenly disappeared. These coefficients are defined, such that

$$N = C_N \frac{1}{2}\rho V^2 S, \qquad T = C_T \frac{1}{2}\rho V^2 S, \qquad M = C_m \frac{1}{2}\rho V^2 S\bar{c}, \quad (5.2.10)$$

$$N_w = C_{N_w} \frac{1}{2}\rho V^2 S, \qquad T_w = C_{T_w} \frac{1}{2}\rho V^2 S, \qquad M_{ac_w} = C_{m_{ac_w}} \frac{1}{2}\rho V^2 S\bar{c}, \quad (5.2.11)$$

$$N_h = C_{N_h} \frac{1}{2}\rho V_h^2 S_h, \qquad T_h = C_{T_h} \frac{1}{2}\rho V_h^2 S_h, \qquad M_{ac_h} = C_{m_{ac_h}} \frac{1}{2}\rho V_h^2 S_h \bar{c}_h. \quad (5.2.12)$$

Here, $\frac{1}{2}\rho V_h^2$ is the average local dynamic pressure on the horizontal tail plain. Also, S_h is the tailplane surface area and $\bar{c}_h$ is the MAC of the tailplane. The quantity $\frac{S_h l_h}{S\bar{c}}$ is known as the **tailplane volume**. And finally, we have defined the **tail length** $l_h = x_h - x_w \approx x_h - x_{cg}$.

5.3 The horizontal tailplane

5.3.1 Important angles

We will now take a closer look at the horizontal tailplane. There are three parameters that describe the configuration of the horizontal tailplane. These parameters are the **effective horizontal tailplane angle of attack** α_h, the **elevator deflection** δ_e and the **elevator trim tab deflection** δ_{t_e}. The three angles are visualized in figure 5.1.

There is one angle which we will examine more closely now. And that is the effective angle of attack α_h. It is different from the angle of attack of the aircraft α. There are two important causes for this. First,

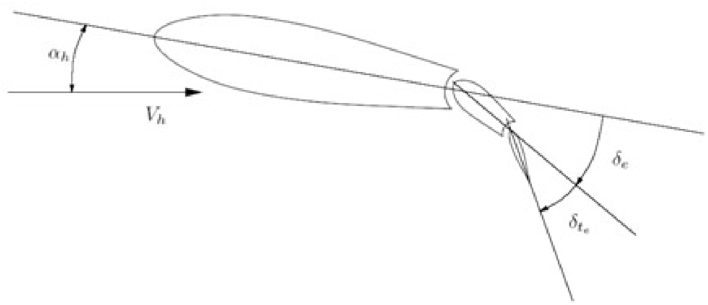

Figure 5.1: The angles of the horizontal tailplane.

the horizontal tail plane has an **incidence angle** i_h, relative to the MAC of the wing. And second, the tailplane experiences downwash, caused by the wing of the aircraft. The **average downwash angle** is denoted by ε. By putting this all together, we find that

$$\alpha_h = \alpha + i_h - \varepsilon. \tag{5.3.1}$$

We can elaborate a bit further on this. The downwash ε mainly depends on α. Linearization thus gives $\varepsilon \approx \frac{d\varepsilon}{d\alpha}(\alpha - \alpha_0)$. It follows that

$$\alpha_h = \left(1 - \frac{d\varepsilon}{d\alpha}\right)(\alpha - \alpha_0) + (\alpha_0 + i_h). \tag{5.3.2}$$

From this follows that the derivate $d\alpha_h/d\alpha$ is given by

$$\frac{d\alpha_h}{d\alpha} = 1 - \frac{d\varepsilon}{d\alpha}. \tag{5.3.3}$$

This derivative is thus generally smaller than 1.

5.3.2 The horizontal tailplane normal force

Let's examine the normal force C_{N_h} of the horizontal tailplane. This is a function of the three angles α_h, δ_e and δ_{t_e}. Applying linearization gives

$$C_{N_h} = C_{N_{h_0}} + \frac{\partial C_{N_h}}{\partial \alpha_h}\alpha_h + \frac{\partial C_{N_h}}{\partial \delta_e}\delta_e + \frac{\partial C_{N_h}}{\partial \delta_{e_t}}\delta_{e_t}. \tag{5.3.4}$$

The effect of the trim tab to the normal force is usually negligible. So, $\partial C_{N_h}/\partial \delta_{e_t} \approx 0$. Also, since most horizontal tailplanes are (nearly) symmetric, we have $C_{N_{h_0}} \approx 0$. This simplifies the above equation to

$$C_{N_h} = \frac{\partial C_{N_h}}{\partial \alpha_h}\alpha_h + \frac{\partial C_{N_h}}{\partial \delta_e}\delta_e = C_{N_{h_\alpha}}\alpha_h + C_{N_{h_\delta}}\delta_e. \tag{5.3.5}$$

Note that we have used a shorter notation in the right part of the above equation. The variables $C_{N_{h_\alpha}}$ and $C_{N_{h_\delta}}$ are quite important for the balance of the control surface. If they are both negative, then the control surface is called **aerodynamically underbalanced**. If, however, they are both positive, then the control surface is called **aerodynamically overbalanced**.

5.3.3 The elevator deflection necessary for equilibrium

We can ask ourselves, what elevator deflection δ_e should we have, to make sure our aircraft is in equilibrium? For that, we examine the moment equation (5.2.9). In this equation are the coefficients C_{N_w} and

C_{N_h}. We can replace these by the linearizations

$$C_{N_w} = C_{N_{w_\alpha}}(\alpha - \alpha_0) \qquad \text{and} \qquad C_{N_h} = C_{N_{h_\alpha}}\alpha_h + C_{N_{h_\delta}}\delta_e. \tag{5.3.6}$$

If we do this, we find that

$$C_m = C_{m_{ac_w}} + C_{N_{w_\alpha}}(\alpha - \alpha_0)\frac{x_{cg} - x_w}{\bar{c}} - \left(C_{N_{h_\alpha}}\alpha_h + C_{N_{h_\delta}}\delta_e\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} = 0. \tag{5.3.7}$$

We can now also substitute the relation (5.3.2) for α_h. Doing this, and working the whole equation out, gives

$$C_m = C_{m_0} + C_{m_\alpha}(\alpha - \alpha_0) + C_{m_{\delta_e}}\delta_e = 0, \tag{5.3.8}$$

where C_{m_α} is known as the **static longitudinal stability** and $C_{m_{\delta_e}}$ is the **elevator effectivity**. Together with the constant C_{m_0}, they are defined as

$$C_{m_0} = C_{m_{ac_w}} - C_{N_{h_\alpha}}(\alpha_0 + i_h)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}, \tag{5.3.9}$$

$$C_{m_\alpha} = C_{N_{w_\alpha}}\frac{x_{cg} - x_w}{\bar{c}} - C_{N_{h_\alpha}}\left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}, \tag{5.3.10}$$

$$C_{m_{\delta_e}} = -C_{N_{h_\delta}}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{5.3.11}$$

We can now solve for δ_e. It is simply given by

$$\delta_e = -\frac{C_{m_0} + C_{m_\alpha}(\alpha - \alpha_0)}{C_{m_{\delta_e}}}. \tag{5.3.12}$$

This is a nice expression. But do remember that we have made several linearizations to derive this equation. The above equation is thus only valid, when all the linearizations are allowed.

6. Longitudinal stability derivatives

We have seen a lot of stability derivatives in previous chapters. However, it would be nice to know their values. We're therefore going to derive some relations for them. In this chapter, we will look at longitudinal stability derivatives. In the next chapter, we'll examine lateral stability derivatives.

6.1 Nominal stability derivatives

6.1.1 Methods of finding the stability derivatives

There are three methods to find the stability derivatives. The first one is of course by performing **flight tests** or **wind tunnel tests**. These tests are, however, quite expensive. An alternative is using **computational fluid dynamics** (CDF). This is usually less expensive, but it still requires a lot of work.

Finally, simple **analytic expressions** can be used. They are usually not very accurate. (Especially not for strange aircraft configurations.) However, they don't require a lot of work. In this chapter, we're going to examine these analytic expressions.

6.1.2 Equations of motion

Before we will find stability derivatives, we first need to re-examine the equations of motion. The symmetric equations of motion, for an airplane in a steady flight, are

$$X = -D\cos\alpha + L\sin\alpha + T_p\cos(\alpha_0 + i_p) = W\sin\gamma_0, \tag{6.1.1}$$

$$Z = -L\cos\alpha - D\sin\alpha - T_p\sin(\alpha_0 + i_p) = -W\cos\gamma_0. \tag{6.1.2}$$

Here, α_0 is the initial angle of attack. (It is now not the zero-lift angle of attack.) Also, α now denotes the deviation from this flight condition. We assume $\alpha_0 + i_p$ is small, so we can use the small angle approximation. If we also non-dimensionalize the above relations, we find that

$$C_X = -C_D\cos\alpha + C_L\sin\alpha + T_c' = \frac{W}{\frac{1}{2}\rho V^2 S}\sin\gamma_0, \tag{6.1.3}$$

$$C_Z = -C_L\cos\alpha - C_D\sin\alpha - T_c'(\alpha_0 + i_p) = -\frac{W}{\frac{1}{2}\rho V^2 S}\cos\gamma_0, \tag{6.1.4}$$

where we have defined

$$T_c' = \frac{T_p}{\frac{1}{2}\rho V^2 S}. \tag{6.1.5}$$

6.1.3 Nominal flight conditions

Let's examine an aircraft flying a steady horizontal flight. We will now try to find the nominal stability derivatives C_{X_0}, C_{Z_0} and C_{m_0}. Since the aircraft is flying horizontally, we have $\alpha = \gamma_0 = 0$. (Remember that α is the deviation from the steady flight.) The relations of the previous paragraph now turn into

$$C_{X_0} = -C_D + T_c' = 0 \qquad \text{and} \qquad C_{Z_0} = -C_L - T_c'(\alpha_0 + i_p) = \frac{W}{\frac{1}{2}\rho V^2 S}. \tag{6.1.6}$$

Finally, from moment equilibrium follows that $C_{m_0} = 0$.

6.2 Velocity stability derivatives

6.2.1 The basic relations

Now let's find the stability derivatives with respect to the velocity. They are C_{X_u}, C_{Z_u} and C_{m_u}. They are very hard to determine experimentally. This is because wind tunnels and flying aircraft can't change their velocity in a very accurate way. Luckily, we can find expressions for them.

Let's start to examine C_{X_u}. We can recall that

$$C_{X_u} = \frac{1}{\frac{1}{2}\rho V S}\frac{\partial X}{\partial V} \qquad \text{and} \qquad X = C_X \frac{1}{2}\rho V^2 S. \tag{6.2.1}$$

(We have used the fact that $\partial V/\partial u \approx 1$.) Taking the derivative of the second relation, with respect to V, gives

$$\frac{\partial X}{\partial V} = C_X \rho V S + \frac{\partial C_X}{\partial V}\frac{1}{2}\rho V^2 S. \tag{6.2.2}$$

Inserting this into the first relation will give

$$C_{X_u} = 2C_X + \frac{\partial C_X}{\partial V} V. \tag{6.2.3}$$

In a similar way, we can find the expressions for C_{Z_u} and C_{m_u}. They are

$$C_{Z_u} = 2C_Z + \frac{\partial C_Z}{\partial V} V \qquad \text{and} \qquad C_{m_u} = 2C_m + \frac{\partial C_m}{\partial V} V. \tag{6.2.4}$$

6.2.2 Rewriting the relations

There are still some terms we don't know in the relations of the previous paragraph. They are C_X, C_Z, C_m, $\partial C_X/\partial V$, $\partial C_Z/\partial V$ and $\partial C_m/\partial V$. How can we rewrite them?

We are considering deviations from the steady horizontal flight. So we can replace C_X by $C_{X_0} = -C_D + T_c'$. Similarly, C_Z is replaced by $C_{Z_0} = -C_L - T_c'(\alpha_0 + i_p)$ and C_m by $C_{m_0} = 0$. That simplifies the equations quite a bit.

The derivatives are a bit harder to rewrite. At a steady horizontal flight, we have $C_X = -C_D + T_c'$ and $C_Z = -C_L - T_c'(\alpha_0 + i_p)$. Differentiating this gives

$$\frac{\partial C_X}{\partial V} = -\frac{\partial C_D}{\partial V} + \frac{\partial T_c'}{\partial V} \qquad \text{and} \qquad \frac{\partial C_Z}{\partial V} = -\frac{\partial C_L}{\partial V} - \frac{\partial T_c'}{\partial V}(\alpha_0 + i_p). \tag{6.2.5}$$

That leaves us with some more derivatives. First, let's examine $\partial C_D/\partial V$, $\partial C_L/\partial V$ and $\partial C_m/\partial V$. To find them, we have to know why the coefficients C_D, C_L and C_m vary with airspeed. These variations are partly caused by changes in Mach number M, changes in thrust T_c' and changes in Reynolds number Re. Although changes in the Reynolds number may be neglected, we do have to consider M and T_c'. This implies that

$$\frac{\partial C_D}{\partial V} = \frac{\partial C_D}{\partial M}\frac{\partial M}{\partial V} + \frac{\partial C_D}{\partial T_c'}\frac{\partial T_c'}{\partial V}, \tag{6.2.6}$$

$$\frac{\partial C_L}{\partial V} = \frac{\partial C_L}{\partial M}\frac{\partial M}{\partial V} + \frac{\partial C_L}{\partial T_c'}\frac{\partial T_c'}{\partial V}, \tag{6.2.7}$$

$$\frac{\partial C_m}{\partial V} = \frac{\partial C_m}{\partial M}\frac{\partial M}{\partial V} + \frac{\partial C_m}{\partial T_c'}\frac{\partial T_c'}{\partial V}. \tag{6.2.8}$$

If we use this, in combination with earlier equations, we will find that

$$C_{X_u} = -2C_D + 2T'_c + \left(1 - \frac{\partial C_D}{\partial T'_c}\right)\frac{dT'_c}{dV}V - \frac{\partial C_D}{\partial M}M, \tag{6.2.9}$$

$$C_{Z_u} = -2C_L - 2T'_c(\alpha_0 + i_p) - \left((\alpha_0 + i_p) + \frac{\partial C_L}{\partial T'_c}\right)\frac{dT'_c}{dV}V - \frac{\partial C_L}{\partial M}M, \tag{6.2.10}$$

$$C_{m_u} = \frac{\partial C_m}{\partial T'_c}\frac{dT'_c}{dV}V + \frac{\partial C_m}{\partial M}M. \tag{6.2.11}$$

6.2.3 The thrust derivative

The equations of the previous paragraph still have a lot of derivatives. We won't go into detail on derivatives with respect T'_c or M. However, we will consider dT'_c/dV. It turns out that we can write this derivative as

$$\frac{dT'_c}{dV} = -k\frac{T'_c}{V}, \tag{6.2.12}$$

where the constant k depends on the flight type. If we have a **gliding flight**, then $T'_c = 0$. Thus also $dT'_c/dV = 0$ and therefore $k = 0$. If we have a **jet-powered aircraft**, then $T_p = T'_c\frac{1}{2}\rho V^2 S = \text{constant}$. From this follows that $k = 2$. Finally, for **propeller-powered aircraft**, we have $T_p V = T'_c\frac{1}{2}\rho V^3 S = \text{constant}$. This implies that $k = 3$.

Let's use the above relation to simplify our equations. The equations for C_{X_u}, C_{Z_u} and C_{m_u} now become

$$C_{X_u} = -2C_D + \left(2 - k\left(1 - \frac{\partial C_D}{\partial T'_c}\right)\right)T'_c - \frac{\partial C_D}{\partial M}M, \tag{6.2.13}$$

$$C_{Z_u} = -2C_L + \left((k-2)(\alpha_0 + i_p) + k\frac{\partial C_L}{\partial T'_c}\right)T'_c - \frac{\partial C_L}{\partial M}M, \tag{6.2.14}$$

$$C_{m_u} = -k\frac{\partial C_m}{\partial T'_c}T'_c + \frac{\partial C_m}{\partial M}M. \tag{6.2.15}$$

When specific data about the type of flight is known, the above equations can be simplified even further. For example, when the flight is at low subsonic velocities, then Mach effects may be neglected. Thus $\partial C_D/\partial M = \partial C_L/\partial M = \partial C_m/\partial M = 0$. In other cases, there are often other simplifications that can be performed.

6.3 Angle of attack stability derivatives

6.3.1 The basic relations for C_{X_α} and C_{Z_α}

We will now try to find relations for C_{X_α}, C_{Z_α} and C_{m_α}. First we examine C_{X_α} and C_{Z_α}. They are defined as

$$C_{X_\alpha} = \frac{1}{\frac{1}{2}\rho V S}\frac{\partial X}{\partial w} = \frac{\partial C_X}{\partial \alpha} \qquad \text{and} \qquad C_{Z_\alpha} = \frac{1}{\frac{1}{2}\rho V S}\frac{\partial Z}{\partial w} = \frac{\partial C_Z}{\partial \alpha}. \tag{6.3.1}$$

If we take the derivative of equations (6.1.3) and (6.1.4), we find that

$$C_{X_\alpha} = -C_{D_\alpha}\cos\alpha + C_D\sin\alpha + C_{L_\alpha}\sin\alpha + C_L\cos\alpha, \tag{6.3.2}$$

$$C_{Z_\alpha} = -C_{L_\alpha}\cos\alpha + C_L\sin\alpha - C_{D_\alpha}\sin\alpha - C_D\cos\alpha. \tag{6.3.3}$$

We are examining an aircraft performing a steady horizontal flight. Thus $\alpha = 0$. This simplifies the above equations to

$$C_{X_\alpha} = C_L - C_{D_\alpha} \qquad \text{and} \qquad C_{Z_\alpha} = -C_{L_\alpha} - C_D \approx -C_{L_\alpha} \approx -C_{N_\alpha}. \tag{6.3.4}$$

In the last part of the above equation, we have used the fact that C_D is much smaller than C_{L_α}.

6.3.2 Rewriting the relation for C_{X_α}

We can try to rewrite the relation for C_{X_α}. To do this, we examine C_{D_α}. Let's assume that the aircraft has a parabolic drag curve. This implies that

$$C_D = C_{D_0} + \frac{C_L^2}{\pi A e}, \qquad \text{which, in turn, implies that} \qquad C_{D_\alpha} = 2\frac{C_{L_\alpha}}{\pi A e} C_L. \tag{6.3.5}$$

If we combine this with the former expression for C_{X_α}, we wind up with

$$C_{X_\alpha} = C_L \left(1 - 2\frac{C_{L_\alpha}}{\pi A e}\right). \tag{6.3.6}$$

6.3.3 The relation for C_{m_α}

In a previous chapter, we have already considered C_{m_α}. After neglecting the effects of many parts of the aircraft, we wound up with

$$C_{m_\alpha} = C_{N_{w_\alpha}} \frac{x_{cg} - x_w}{\bar{c}} - C_{N_{h_\alpha}} \left(1 - \frac{d\varepsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{6.3.7}$$

6.4 Pitch rate stability derivatives

6.4.1 The reasons behind the changing coefficients

We will now try to find C_{X_q}, C_{Z_q} and C_{m_q}. Luckily, C_X doesn't get influenced a lot by q. So it is usually assumed that $C_{X_q} = 0$. That saves us some work. We now only need to find C_{Z_q} and C_{m_q}. They are defined as

$$C_{Z_q} = \frac{1}{\frac{1}{2}\rho V S\bar{c}} \frac{\partial Z}{\partial q} = \frac{\partial C_Z}{\partial \frac{q\bar{c}}{V}} \qquad \text{and} \qquad C_{m_q} = \frac{1}{\frac{1}{2}\rho V S\bar{c}^2} \frac{\partial M}{\partial q} = \frac{\partial C_m}{\partial \frac{q\bar{c}}{V}}. \tag{6.4.1}$$

To find C_{Z_q} and C_{m_q}, we first have to understand some theory behind rotations. Why do the coefficients change, when the aircraft rotates? This is because the effective angle of attack changes. Imagine an aircraft with its nose pitching upward. The tailplane of the aircraft is thus pitching downward. Now imagine you're sitting on the tailplane. As seen from the tailplane, it looks like the flow of air is coming upward. This means that the tailplane experiences a bigger angle of attack.

To find the exact value of the change in angle of attack $\Delta\alpha$, we examine the **center of rotation**. This is the point about which the aircraft appears to be rotating. The center of rotation lies on the Z_s-axis. The apparent rotation itself is performed with a radius R, which is given by

$$R = \frac{V}{q}. \tag{6.4.2}$$

The change in angle of attack $\Delta\alpha$, at any point x on the airplane, is then given by

$$\Delta\alpha = \frac{x - x_{cg}}{R} = \frac{x - x_{cg}}{\bar{c}} \frac{q\bar{c}}{V}. \tag{6.4.3}$$

6.4.2 The changing coefficients

We know that the apparent angle of attack changes across the aircraft. This is especially important for the horizontal tailplane. In fact, the change in angle of attack of the tailplane is given by

$$\Delta\alpha_h = \frac{x_h - x_{cg}}{\bar{c}} \frac{q\bar{c}}{V} \approx \frac{l_h}{\bar{c}} \frac{q\bar{c}}{V}. \tag{6.4.4}$$

This change in angle of attack causes the normal force of the tailplane to change. In fact, it changes by an amount

$$\Delta C_{N_h} = C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \Delta\alpha_h = C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \frac{q\bar{c}}{V}. \tag{6.4.5}$$

Similarly, the change of the moment is given by

$$\Delta C_m = -C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \Delta\alpha_h = -C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h^2}{S\bar{c}^2} \frac{q\bar{c}}{V}. \tag{6.4.6}$$

We know that $C_{Z_q} = \partial C_Z / \partial \frac{q\bar{c}}{V}$ and $C_{m_q} = \partial C_m / \partial \frac{q\bar{c}}{V}$. By using this, we can find the contributions of the horizontal tailplane to C_{z_q} and C_{m_q}. They are

$$\left(C_{Z_q}\right)_h = -C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \qquad \text{and} \qquad \left(C_{m_q}\right)_h = -C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h^2}{S\bar{c}^2}. \tag{6.4.7}$$

(The minus sign in the left part appeared, because C_N is defined upward, while C_Z is defined downward.) There is, however, one small problem. The aircraft doesn't consist of only a horizontal tailplane. It also has various other parts. But it is very difficult to calculate the effects of all these parts. For that reason, we make an estimate. We say that the contribution of the full aircraft C_{Z_q} is twice the contribution of the horizontal tailplane $\left(C_{Z_q}\right)_h$. This implies that

$$C_{Z_q} = 2\left(C_{Z_q}\right)_h = 2C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{6.4.8}$$

For C_{m_q}, we apply the same trick. But instead of a factor 2, a factor between 1.1 to 1.2 should now be used, depending on the aircraft. We thus get

$$C_{m_q} = (1.1 \sim 1.2)\left(C_{m_q}\right)_h = -(1.1 \sim 1.2) C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h^2}{S\bar{c}^2}. \tag{6.4.9}$$

6.5 Other longitudinal stability derivatives

6.5.1 Vertical acceleration stability derivatives

We now examine $C_{Z_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$. (We assume $C_{X_{\dot{\alpha}}} = 0$.) To do this, we look at the horizontal tailplane. During a steady flight, it has an effective angle of attack

$$\alpha_h = \alpha - \varepsilon + i_h = \alpha - \frac{d\varepsilon}{d\alpha}\alpha + i_h. \tag{6.5.1}$$

Now let's suppose that the aircraft experiences a change in angle of attack. This causes the downwash angle ε of the wing to change. A time $\Delta t = l_h/V$ later will this change be experienced by the horizontal tailplane. In other words, the downwash $\varepsilon(t)$ at time t depends on the angle of attack $\alpha(t - \Delta t)$ at time $t - \Delta t$. A linear approximation of $\alpha(t - \Delta t)$ is given by

$$\alpha(t - \Delta t) = \alpha(t) - \dot{\alpha}\Delta t. \tag{6.5.2}$$

By using this, we find that the downwash is given by

$$\varepsilon(t) = \frac{d\varepsilon}{d\alpha}\alpha(t - \Delta t) = \frac{d\varepsilon}{d\alpha}\alpha(t) - \frac{d\varepsilon}{d\alpha}\dot{\alpha}\frac{l_h}{V}. \tag{6.5.3}$$

This implies that the effective angle of attack is given by

$$\alpha_h = \alpha - \frac{d\varepsilon}{d\alpha}\alpha + \frac{d\varepsilon}{d\alpha}\dot{\alpha}\frac{l_h}{V} + i_h. \tag{6.5.4}$$

The change in effective angle of attack is

$$\Delta\alpha_h = \frac{d\varepsilon}{d\alpha}\frac{l_h}{\bar{c}}\frac{\dot{\alpha}\bar{c}}{V}. \tag{6.5.5}$$

We now have enough data to find the coefficients $C_{Z_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$. We know that

$$\Delta C_Z = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h}{S}\Delta\alpha_h = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}\frac{d\varepsilon}{d\alpha}\frac{\dot{\alpha}\bar{c}}{V}, \tag{6.5.6}$$

$$\Delta C_m = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}\Delta\alpha_h = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h^2}{S\bar{c}^2}\frac{d\varepsilon}{d\alpha}\frac{\dot{\alpha}\bar{c}}{V}. \tag{6.5.7}$$

The coefficients $C_{Z_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$ are now given by

$$C_{Z_{\dot{\alpha}}} = \frac{1}{\frac{1}{2}\rho S\bar{c}}\frac{\partial Z}{\partial\dot{w}} = \frac{\partial C_Z}{\partial\frac{\dot{\alpha}\bar{c}}{V}} = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}\frac{d\varepsilon}{d\alpha}, \tag{6.5.8}$$

$$C_{m_{\dot{\alpha}}} = \frac{1}{\frac{1}{2}\rho S\bar{c}^2}\frac{\partial M}{\partial\dot{w}} = \frac{\partial C_m}{\partial\frac{\dot{\alpha}\bar{c}}{V}} = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h^2}{S\bar{c}^2}\frac{d\varepsilon}{d\alpha}. \tag{6.5.9}$$

6.5.2 Elevator angle stability derivatives

The last stability derivatives we will consider in this chapter are $C_{X_{\delta_e}}$, $C_{Z_{\delta_e}}$ and $C_{m_{\delta_e}}$. Usually C_X doesn't vary a lot with δ_e, so we assume that $C_{X_{\delta_e}} = 0$. But what about $C_{Z_{\delta_e}}$? Well, this one is given by

$$C_{Z_{\delta_e}} = -C_{N_{h_{\delta_e}}}\left(\frac{V_h}{V}\right)^2\frac{S_h}{S}. \tag{6.5.10}$$

Finally there is $C_{m_{\delta_e}}$. We can find that it is

$$C_{m_{\delta_e}} = C_{Z_{\delta_e}}\frac{x_h - x_{cg}}{\bar{c}} \approx -C_{N_{\delta_e}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}. \tag{6.5.11}$$

The coefficient $C_{Z_{\delta_e}}$ usually isn't very important. However, $C_{m_{\delta_e}}$ is very important. This is because the whole goal of an elevator is to apply a moment to the aircraft.

6.5.3 Effects of moving the center of gravity

We have one topic left to discuss. What happens when the CG moves from position 1 to position 2? In this case, several coefficients change. This goes according to

$$C_{m_{\alpha_2}} = C_{m_{\alpha_1}} - C_{Z_\alpha}\frac{x_{cg_2} - x_{cg_1}}{\bar{c}}, \tag{6.5.12}$$

$$C_{Z_{q_2}} = C_{Z_{q_1}} - C_{Z_\alpha}\frac{x_{cg_2} - x_{cg_1}}{\bar{c}}, \tag{6.5.13}$$

$$C_{m_{q_2}} = C_{m_{q_1}} - \left(C_{Z_{q_1}} + C_{m_{\alpha_1}}\right)\frac{x_{cg_2} - x_{cg_1}}{\bar{c}} + C_{Z_\alpha}\left(\frac{x_{cg_2} - x_{cg_1}}{\bar{c}}\right)^2, \tag{6.5.14}$$

$$C_{m_{\dot{\alpha}_2}} = C_{m_{\dot{\alpha}_1}} - C_{Z_{\dot{\alpha}}}\frac{x_{cg_2} - x_{cg_1}}{\bar{c}}. \tag{6.5.15}$$

7. Lateral stability derivatives

In the previous chapter, we found relations for the longitudinal stability derivatives. Now we'll examine the lateral stability derivatives.

7.1 Sideslip angle stability derivatives

7.1.1 Horizontal forces

We start by examining derivatives with respect to the **sideslip angle** β. This angle is defined as

$$\beta = \arcsin\left(\frac{v}{V}\right) \approx \frac{v}{V}. \tag{7.1.1}$$

We will now examine C_{Y_β}. Let's examine an aircraft with a sideslip angle β. This sideslip angle causes a horizontal force Y on the aircraft. The most important contributors to this horizontal force are the fuselage and the vertical tailplane.

First let's examine the vertical tailplane. Luckily, this tailplane has a lot of analogies with the horizontal tailplane, so we can use some short cuts. For example, the force acting on the vertical tailplane is given by

$$\left(C_{Y_\beta}\right)_v = C_{Y_{v_\alpha}} \frac{d\alpha_v}{d\beta} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S}, \tag{7.1.2}$$

where S_v is the vertical tailplane surface area and V_v is the average velocity over it. Also, $C_{Y_{v_\alpha}} = \frac{\partial C_{Y_v}}{\partial \alpha_v}$.

Let's take a closer look at the **effective angle of attack of the tailplane** α_v. It's not equal to β. This is because the fuselage also alters the flow by an angle σ. (σ is similar to the downwash ε for the horizontal tailplane.) The vertical tailplane thus has an angle of attack of $\alpha_v = -(\beta - \sigma)$. (The minus is present due to sign convention.) Inserting this relation into the above equation gives

$$\left(C_{Y_\beta}\right)_v = -C_{Y_{v_\alpha}} \left(1 - \frac{d\sigma}{d\beta}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_v}{S}. \tag{7.1.3}$$

Usually, most terms in the above equation are known. Only $d\sigma/d\beta$ is still a bit of a mystery. It is very hard to determine. However, it usually is negative. (So $d\sigma/d\beta < 0$.)

Next to the tailplane contribution, there is usually also a contribution by the fuselage. However, we don't go into depth on that here.

7.1.2 Rolling momemts

Now let's examine the so-called **effective dihedral** C_{l_β}. The coefficient C_l was defined as

$$C_l = \frac{L}{\frac{1}{2}\rho V^2 b}. \tag{7.1.4}$$

It is important to note that L is not the lift. It is the moment about the X axis. C_l is thus not the lift coefficient either.

The effective dihedral C_{l_β} mostly depends on the wing set-up. Both the **wing-dihedral** Γ and the **sweep angle** Λ strongly effect C_l. (The wing-dihedral Γ is the angle with which the wings have been tilted upward, when seen from the fuselage.)

First let's examine an aircraft with a wing-dihedral Γ. We suppose that the aircraft is sideslipping to the right. From the aircraft, it now appears as if part of the flow is coming from the right. This flow 'crouches' under the right wing, pushing it more upward. However, it flows over the left wing, pushing that one downward a bit. This thus causes the aircraft to roll to the left.

To find more info about the moment caused by the wing-dihedral, we need to examine the new angle of attacks of the wings α_{w_l} and α_{w_r}. By using small angle approximations, we can find that

$$\alpha_{w_l} \approx \alpha - \beta\Gamma \qquad \text{and} \qquad \alpha_{w_r} \approx \alpha + \beta\Gamma. \tag{7.1.5}$$

The changes in the angles of attack are thus $\Delta\alpha_{w_l} = -\beta\Gamma$ and $\Delta\alpha_{w_r} = \beta\Gamma$. So the moment caused by the wing-dihedral is approximately linearly dependend on both β and Γ. (We thus have $C_{l_\beta} \sim \Gamma$.)

Second, we look at an aircraft with a wing sweep angle Λ. The lift of a wing strongly depends on the flow velocity perpendicular to the leading edge. Again, we suppose that part of the flow is coming in from the right. This causes the flow to be more perpendicular w.r.t. to the right wing leading edge, thus increasing the lift. However, the flow is more parallel w.r.t. the leading edge of the left wing. The left wing thus has reduced lift. It can be shown that the change in lift for the aircraft, due to a sweep angle Λ, is

$$\Delta L = C_L \frac{1}{2}\rho V^2 \frac{S}{2}\left(\cos^2(\Lambda - \beta) - \cos^2(\Lambda + \beta)\right) \approx C_L \frac{1}{2}\rho V^2 S \sin(2\Lambda\beta). \tag{7.1.6}$$

The rightmost part of the equation is an approximation. It only works for small values of β. The above equation shows that the lift more or less linearly depends on Λ and β. It can be shown that the same holds for the moment C_l. The effective dihedral C_{l_β} is thus proportional to Λ.

Next to the wing, also the horizontal tailplane and the fuselage effect C_{l_β}. However, we won't examine these effects.

7.1.3 Yawing moments

The stability derivative C_{n_β} is called the **static directional stability**. (It's also known as the **Weathercock stability**.) It is about just as important as C_{m_α}. It can be shown that, if C_{n_β} is positive, then the aircraft is stable for yawing motions. However, if C_{n_β} is negative, then the aircraft is unstable for yawing motions.

Naturally, we want to have $C_{n_\beta} > 0$. Luckily, the wings and the horizontal tailplane have a slightly positive effect on C_{n_β}. However, the fuselage causes C_{n_β} to decrease. To compensate for this, a vertical tailplane is used, strongly increasing C_{n_β}.

Let's examine the effects of this tailplane. You may remember that the normal force on it was

$$\left(C_{Y_\beta}\right)_v = -C_{Y_{v_\alpha}}\left(1 - \frac{d\sigma}{d\beta}\right)\left(\frac{V_v}{V}\right)^2 \frac{S_v}{S}. \tag{7.1.7}$$

This normal force causes a moment

$$\left(C_{n_\beta}\right)_v = -\left(C_{Y_\beta}\right)_v\left(\frac{z_v - z_{cg}}{b}\sin\alpha_0 + \frac{x_v - x_{cg}}{b}\cos\alpha_0\right). \tag{7.1.8}$$

We can usually assume α_0 to be small. (Thus $\cos\alpha_0 \approx 1$.) Also, $\frac{z_v - z_{cg}}{b}\sin\alpha_0$ is usually quite small, compared to the other term, so we neglect it. If we also use the **tail length** of the vertical tailplane $l_v = x_v - x_{cg}$, we can rewrite the above equation to

$$\left(C_{n_\beta}\right)_v = C_{Y_{v_\alpha}}\left(1 - \frac{d\sigma}{d\beta}\right)\left(\frac{V_v}{V}\right)^2 \frac{S_v l_v}{Sb}. \tag{7.1.9}$$

From this, the correspondence to C_{m_α} again becomes clear. To emphasize this, we once more show the equation for the horizontal tailplane contribution to C_{m_α}. Rather similar to $\left(C_{n_\beta}\right)_v$, it was given by

$$\left(C_{m_\alpha}\right)_h = -C_{N_{h_\alpha}}\left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{7.1.10}$$

7.2 Roll rate stability derivatives

7.2.1 Horizontal forces

It is time to investigate the effects of roll. In other words, we will try to find the stability derivatives C_{Y_p}, C_{l_p} and C_{n_p}. (Of these three, C_{l_p} is the most important.) First we examine C_{Y_p}. It is defined such that

$$Y_p = C_{Y_p} \frac{pb}{2V} \frac{1}{2} \rho V^2 S. \tag{7.2.1}$$

The only part having a more or less significant contribution to C_{Y_p} is the vertical tailplane. Let's examine a rolling aircraft. Due to this rolling, the vertical tailplane is moving horizontally. It will therefore get an effective angle of attack. This causes a horizontal force. A positive roll rate gives a negative horizontal force. C_{Y_p} is thus negative.

However, C_{Y_p} is usually rather small. For this reason it is often neglected. So we say that $C_{Y_p} \approx 0$.

7.2.2 Rolling moments

Now we will try to find C_{l_p}. Again, we examine a rolling aircraft. One wing of the aircraft goes up, while the other one goes down. This motion changes the effective angle of attack and thus also the lift of the wings. The upward going wing will get a lower lift, while the downward moving wing will experience a bigger amount of lift. The wing forces thus cause a moment opposite to the rolling motion. This means that C_{l_p} is highly negative. It also implies that the rolling motion is very strongly damped. (We will see this again, when examining the aperiodic roll in chapter 10.)

We can also investigate the actual effects of the rolling motion. To do this, we examine a chord at a distance y from the fuselage. This chord will have an additional vertical velocity of py. The change in angle of attack of this chord thus is

$$\Delta\alpha = \frac{py}{V} = \frac{pb}{2V} \frac{y}{b/2}. \tag{7.2.2}$$

So a chord that is far away from the fuselage will experience a big change in angle of attack. The change in lift is therefore biggest for these chords. These chords also have a relatively big distance to the CG of the aircraft. For this reason, they will significantly effect the resulting moment.

Other parts of the aircraft may also influence C_{l_p} slightly. However, their influence is very small, compared to the effects of the wings. The contributions of the other parts are therefore neglected.

7.2.3 Yawing moments

To find C_{n_p}, we again examine a rolling aircraft. The rolling of the aircraft has two important effects.

First, we look at the vertical tailplane. As was discussed earlier, this tailplane will move. It thus has an effective angle of attack, and therefore a horizontal force. This horizontal force causes the aircraft to yaw. A positive rolling motion causes a positive yawing moment. The vertical tailplane thus has a positive contribution to C_{n_p}. (So $\left(C_{n_p}\right)_v > 0$.)

But now let's look at the wings. Let's suppose that the aircraft is rolling to the right. For the right wing, it then appears as if the flow comes (partially) from below. The lift is per definition perpendicular to the direction of the incoming flow. The lift vector is thus tilted forward. Part of this lift causes the aircraft to yaw to the left. The opposite happens for the left wing: The lift vector is tilted backward. Again, this causes a yawing moment to the left. (This effect is known as **adverse yaw**.) So we conclude that, due to the wings, a positive rolling motion results in a negative yawing moment. We thus have $\left(C_{n_p}\right)_w < 0$.

For most normal flights, the effects of the vertical stabilizer are a bit bigger than the effects of the wing. We thus have $C_{n_p} > 0$. However, high roll rates and/or high angles of attack increase the effect of the wings. In this case, we will most likely have $C_{n_p} < 0$.

7.3 Yaw rate stability derivatives

7.3.1 Horizontal forces

In this part, we'll try to find the stability derivatives C_{Y_r}, C_{l_r} and C_{n_r}. We start with the not very important coefficient C_{Y_r}. Let's examine a yawing aircraft. Due to the yawing moment, the vertical tailplane moves horizontally. Because of this, its effective angle of attack will change by

$$\Delta\alpha_v = \frac{rl_v}{V} = \frac{rb}{2V}\frac{l_v}{b/2}. \tag{7.3.1}$$

The contribution of the tailplane to C_{Y_r} is now given by

$$\left(C_{Y_r}\right)_v = 2C_{Y_{v_\alpha}}\left(\frac{V_v}{V}\right)^2\frac{S_v l_v}{Sb}. \tag{7.3.2}$$

The contribution is positive, so $\left(C_{Y_r}\right)_v > 0$. Next to the vertical tailplane, there are also other parts influencing C_{Y_r}. Most parts have a negative contribution to C_{Y_r}. However, none of these contributions are as big as $\left(C_{Y_r}\right)_v$. The stability derivative C_{Y_r} is therefore still positive. It is only slightly smaller than $\left(C_{Y_r}\right)_v$.

7.3.2 Rolling moments

We will now examine C_{l_r}. There are two important contributions to C_{l_r}. They come from the vertical tailplane and the wings.

First we examine the vertical tailplane. We just saw that a yawing motion causes a horizontal force on the vertical tailplane. This horizontal force causes a moment

$$\left(C_{l_r}\right)_v = \left(C_{Y_r}\right)_v\left(\frac{z_v - z_{cg}}{b}\cos\alpha_0 - \frac{x_v - x_{cg}}{b}\sin\alpha_0\right). \tag{7.3.3}$$

A positive yawing motion results in a positive moment. We thus have $\left(C_{l_r}\right)_v > 0$.

Now let's examine the wings. Because of the yawing motion, one wing will move faster, while the other wing will move slower. This causes the lift on one wing to increase, while it will decrease on the other wing. This results in a rolling moment.

Sadly, it's rather hard to find an equation for the moment caused by the wings. So we won't examine that any further. However, it is important to remember that a positive yawing motion causes a positive rolling moment. We thus have $\left(C_{l_r}\right)_w > 0$. The total coefficient C_{l_r} is then, of course, also positive.

7.3.3 Yawing moments

Finally, we examine C_{n_r}. The most important contribution comes from the vertical tailplane. We know that a yawing motion causes a horizontal force on the vertical tailplane. This force is such that it damps the yawing motion. The contribution $\left(C_{n_r}\right)_v$ is thus very highly negative. In fact, it is given by

$$\left(C_{n_r}\right)_v = -\left(C_{Y_r}\right)_v\frac{l_v}{b} = -2C_{Y_{v_\alpha}}\left(\frac{V_v}{V}\right)^2\frac{S_v l_v^2}{Sb^2}. \tag{7.3.4}$$

The vertical tailplane is about the only part seriously effecting the coefficient C_{n_r}. Sometimes also the fuselage effects it. This effect is also negative. (So $(C_{n_r})_f < 0$.) The coefficient C_{n_r} itself is thus also very strongly negative. This implies that the yawing motion is highly damped.

7.4 Other lateral stability derivatives

7.4.1 Aileron deflections

Let's consider the ailerons. The aileron deflection δ_a is defined as

$$\delta_a = \delta_{a_{right}} - \delta_{a_{left}}. \tag{7.4.1}$$

A deflection of the ailerons causes almost no change in horizontal forces. We thus have $C_{Y_{\delta_a}} = 0$. The so-called **aileron effectiveness** $C_{l_{\delta_a}}$ is, of course, not negligible. (Causing moments about the X axis is what ailerons are for.) The coefficient $C_{n_{\delta_a}}$ usually isn't negligible either. By the way, the moments caused by an aileron deflection are given by

$$L = C_{l_{\delta_a}} \delta_a \frac{1}{2} \rho V^2 S b \qquad \text{and} \qquad N = C_{n_{\delta_a}} \delta_a \frac{1}{2} \rho V^2 S b. \tag{7.4.2}$$

$C_{l_{\delta_a}}$ is negative. A positive aileron deflection causes a negative rolling moment. $C_{n_{\delta_a}}$ is, however, positive. So a positive aileron deflection causes positive yaw.

7.4.2 Rudder deflections

The rudder stability derivatives are $C_{Y_{\delta_r}}$, $C_{l_{\delta_r}}$ and $C_{n_{\delta_r}}$. The forces and moments caused by a rudder deflection are given by

$$Y = C_{Y_{\delta_r}} \delta_r \frac{1}{2} \rho V^2 S, \qquad L = C_{l_{\delta_r}} \delta_r \frac{1}{2} \rho V^2 S b. \quad \text{and} \quad N = C_{n_{\delta_r}} \delta_r \frac{1}{2} \rho V^2 S b. \tag{7.4.3}$$

The coefficient $C_{Y_{\delta_r}}$ is given by

$$C_{Y_{\delta_r}} = C_{Y_{v_\delta}} \left(\frac{V_v}{V} \right)^2 \frac{S_v}{S}. \tag{7.4.4}$$

The coefficient $C_{l_{\delta_r}}$ is then given by

$$C_{l_{\delta_r}} = C_{Y_{\delta_r}} \left(\frac{z_v - z_{cg}}{b} \cos \alpha_0 - \frac{x_v - x_{cg}}{b} \sin \alpha_0 \right). \tag{7.4.5}$$

$C_{l_{\delta_r}}$ is positive. This means that a positive rudder deflection causes a positive rolling moment. This effect is generally not desirable. Especially if $z_v - z_{cg}$ is big, measures are often taken to reduce this effect.

Finally, the **rudder effectiveness** $C_{n_{\delta_r}}$ is given by

$$C_{n_{\delta_r}} = -C_{Y_{\delta_r}} \frac{l_v}{b}. \tag{7.4.6}$$

This coefficient is negative. A positive rudder deflection thus causes a negative yawing moment.

7.4.3 Spoiler deflections

The last things we examine are the spoilers. Spoilers are often used in high-speed aircraft to provide roll control. A spoiler deflection δ_s on the left wing is defined to be positive. Due to this definition, we have $C_{l_{\delta_s}} < 0$ and $C_{n_{\delta_s}} < 0$. Of these two, the latter is the most important.

8. Longitudinal stability and control

In this chapter, we will start to investigate the stability of the entire aircraft. This can be split up into two parts: longitudinal and lateral stability. In this chapter, we will only look at longitudinal stability.

8.1 Stick fixed longitudinal stability

8.1.1 Effects of the wing and the tail on stability

To start our investigation in the stability of an aircraft, we reexamine the moment equation. In an earlier chapter, we found that

$$C_m = C_{m_{ac}} + C_{N_{w_\alpha}}(\alpha - \alpha_0)\frac{x_{cg} - x_w}{\bar{c}} - C_{N_h}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} = 0, \tag{8.1.1}$$

where C_{N_h} is given by

$$C_{N_h} = C_{N_{h_\alpha}}\left((\alpha - \alpha_0)\left(1 - \frac{d\varepsilon}{d\alpha}\right) + (\alpha_0 + i_h)\right) + C_{N_{h_{\delta_e}}}\delta_e. \tag{8.1.2}$$

We can also rewrite the moment equation to $C_m = C_{m_w} + C_{m_h}$. In this equation, C_{m_w} is the contribution due to the wings. Similarly, C_{m_h} is the contribution from the horizontal tailplane. They are both given by

$$C_{m_w} = C_{m_{ac}} + C_{N_{w_\alpha}}(\alpha - \alpha_0)\frac{x_{cg} - x_w}{\bar{c}} \qquad \text{and} \qquad C_{m_h} = -C_{N_h}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{8.1.3}$$

Taking a derivative of the moment equation will give us $C_{m_\alpha} = C_{m_{\alpha_w}} + C_{m_{\alpha_h}}$, where

$$C_{m_{\alpha_w}} = C_{N_{w_\alpha}}\frac{x_{cg} - x_w}{\bar{c}} \qquad \text{and} \qquad C_{m_{\alpha_h}} = -C_{N_{h_\alpha}}\left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{8.1.4}$$

To achieve stability for our aircraft, we should have $C_{m_\alpha} < 0$. Usually, the wing is in front of the CG. We thus have $x_{cg} - x_w > 0$ and also $C_{m_{\alpha_w}} > 0$. The wing thus destabilizes the aircraft. Luckily, the horizontal tailplane has a stabilizing effect. This is because $C_{m_{\alpha_h}} < 0$. To achieve stability, the stabilizing effect of the tailplane should be bigger than the destabilizing effect of the wings. We should thus have

$$|C_{m_{\alpha_w}}| < |C_{m_{\alpha_h}}|. \tag{8.1.5}$$

8.1.2 Effects of the center of gravity on stability

We will now examine the effects of the CG on the stability. To do this, we suppose x_{cg} increases (the CG moves to the rear). However, the other parameters (including δ_e) stay constant. The movement of the CG causes C_{m_α} to increase. At a certain point, we will reach $C_{m_\alpha} = 0$. When the CG moves beyond this position, the aircraft becomes unstable.

Let's examine the point at which $C_{m_\alpha} = 0$. We remember, from a previous chapter, that this point is called the neutral point. And, because the stick deflection is constant (δ_e is constant), we call this point the **stick fixed neutral point**. Its x coordinate is denoted by $x_{n_{fix}}$. To find it, we can use

$$C_{m_\alpha} = C_{N_{w_\alpha}}\frac{x_{n_{fix}} - x_w}{\bar{c}} + C_{N_{h_\alpha}}\left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}\frac{x_{n_{fix}} - x_h}{\bar{c}}. \tag{8.1.6}$$

After some mathematical trickery, we can find the position of the stick fixed neutral point, with respect to the wing. It is given by

$$\frac{x_{n_{fix}} - x_w}{\bar{c}} = \frac{C_{N_{h_\alpha}}}{C_{N_\alpha}} \left(1 - \frac{d\varepsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{8.1.7}$$

From this, we can also derive the position of the stick fixed neutral point, with respect to the aircraft CG. This is given by

$$C_{m_\alpha} = C_{N_\alpha} \frac{x_{cg} - x_{n_{fix}}}{\bar{c}}. \tag{8.1.8}$$

The quantity $\frac{x_{cg} - x_{n_{fix}}}{\bar{c}}$ is known as the **(stick fixed) stability margin**. It is an indication of how much the CG can move, before the aircraft becomes unstable.

8.1.3 The elevator trim curve

Now let's examine the effects of the elvator deflection δ_e. We know from a previous chapter that the elevator deflection necessary to keep the aircraft in equilibrium is

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}} \left(C_{m_0} + C_{m_\alpha} (\alpha - \alpha_0)\right). \tag{8.1.9}$$

δ_e depends on α. To see how, we plot δ_e versus α. We usually do this, such that the y axis is reversed. (Positive δ_e appear below the horizontal axis.) Now we examine the slope of this graph. It is given by

$$\frac{d\delta_e}{d\alpha} = -\frac{C_{m_\alpha}}{C_{m_{\delta_e}}}. \tag{8.1.10}$$

We always have $C_{m_{\delta_e}} < 0$. To make sure we have $C_{m_\alpha} < 0$ as well, we should have $d\delta_e/d\alpha < 0$. The line in the δ_e, α graph should thus go upward as α increases. (Remember that we have reversed the y axis of the graph!)

δ_e also depends on the aircraft velocity V. To see how, we will rewrite equation (8.1.9). By using $C_N \approx C_{N_\alpha} (\alpha - \alpha_0) \approx \frac{W}{\frac{1}{2}\rho V^2 S}$, we find that

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}} \left(C_{m_0} + \frac{C_{m_\alpha}}{C_{N_\alpha}} \frac{W}{\frac{1}{2}\rho V^2 S}\right). \tag{8.1.11}$$

We can now plot δ_e against V. (Again, we reverse the δ_e axis.) We have then created the so-called **elevator trim curve**. Its slope is given by

$$\frac{d\delta_e}{dV} = \frac{4W}{\rho V^3 S} \frac{1}{C_{m_{\delta_e}}} \frac{C_{m_\alpha}}{C_{N_\alpha}}. \tag{8.1.12}$$

To have $C_{m_\alpha} < 0$, we should have $d\delta_e/dV > 0$. The line in the graph should thus go downward. Also, if you want to fly faster in a stable aircraft, you should push your stick forward.

8.2 Stick free longitudinal stability

8.2.1 The stick free elevator deflection

Previously, we have assumed that δ_e is constant. The pilot has his stick fixed. But what will happen if the pilot releases his stick? It would be nice if the aircraft remains stable as well.

Let's suppose the pilot releases the stick. In that case, aerodynamic force will give the elevator a certain **stick free elevator deflection** $\delta_{e_{free}}$. To find $\delta_{e_{free}}$, we examine the moments H_e about the elevator hinge point. (Or, to be more precise, we look at the non-dimensional version C_{h_e}.) Contributing to this hinge moment are the horizontal tailplane, the elevator and the trim tab. By using a linearization, we find that

$$C_{h_{e_{free}}} = C_{h_\alpha}\alpha_h + C_{h_\delta}\delta_{e_{free}} + C_{h_{\delta_t}}\delta_{t_e} = 0. \tag{8.2.1}$$

It follows that the stick free elevator deflection is

$$\delta_{e_{free}} = -\frac{C_{h_\alpha}}{C_{h_\delta}}\alpha_h - \frac{C_{h_{\delta_t}}}{C_{h_\delta}}\delta_{t_e}. \tag{8.2.2}$$

From this, we can also derive that

$$\left(\frac{d\delta_e}{d\alpha}\right)_{free} = -\frac{C_{h_\alpha}}{C_{h_\delta}}\left(1 - \frac{d\varepsilon}{d\alpha}\right). \tag{8.2.3}$$

The elevator deflection thus changes as the angle of attack is changed.

8.2.2 Differences in the moment due to the stick free evelator

The free elevator deflection effects the contribution C_{m_h} of the horizontal tailplane to the moment C_m. Let's investigate this. We can remember that

$$C_{m_h} = -\left(C_{N_{h_\alpha}}\alpha_h + C_{N_{h_\delta}}\delta_e\right)\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}. \tag{8.2.4}$$

We now substitute δ_e by $\delta_{e_{free}}$. If we also differentiate with respect to α, and work things out, we will get

$$C_{m_{\alpha_{h_{free}}}} = -\left(C_{N_{h_\alpha}} - C_{N_{h_\delta}}\frac{C_{h_\alpha}}{C_{h_\delta}}\right)\left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}. \tag{8.2.5}$$

If we compare this equation to the right side of equation (8.1.4), we see that only $C_{N_{h_\alpha}}$ has changed. In fact, we can define

$$C_{N_{h_{\alpha_{free}}}} = C_{N_{h_\alpha}} - C_{N_{h_\delta}}\frac{C_{h_\alpha}}{C_{h_\delta}}. \tag{8.2.6}$$

If we use $C_{N_{h_{\alpha_{free}}}}$, instead of $C_{N_{h_\alpha}}$, then our stability analysis is still entirely valid.

Let's take a closer look at the differences between $C_{N_{h_{\alpha_{free}}}}$ and $C_{N_{h_\alpha}}$. This difference is the term $C_{N_{h_\delta}}\frac{C_{h_\alpha}}{C_{h_\delta}}$. We know that $C_{N_{h_\delta}} > 0$. The term C_{h_δ} is interesting. If it would be positive, then it can be shown that the elevator position is unstable. So, we have to have $C_{h_\delta} < 0$. Finally there is C_{h_α}. This term can be either positive or negative. If it is positive ($C_{h_\alpha} > 0$), then the stick free aircraft will be more stable than the stick fixed aircraft. If, however, it is negative ($C_{h_\alpha} < 0$), then it will be less stable, or possibly even unstable.

8.2.3 The stick free neutral point

Let's find the **stick free neutral point** $x_{n_{free}}$. Finding $x_{n_{free}}$ goes similar to finding $x_{n_{fix}}$. In fact, we can adjust equations (8.1.7) and (8.1.8) to

$$\frac{x_{n_{free}} - x_w}{\bar{c}} = \frac{C_{N_{h_{\alpha_{free}}}}}{C_{N_\alpha}}\left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}, \tag{8.2.7}$$

$$C_{m_{\alpha_{free}}} = C_{N_{\alpha_{free}}} \frac{x_{cg} - x_{n_{free}}}{\bar{c}}. \tag{8.2.8}$$

In this equation, we have $C_{N_{\alpha_{free}}} \approx C_{N_\alpha}$. This is because the elevator has a negligible influence on C_{N_α}, compared to the influence of the wing.

We can also find the position of the stick free neutral point, with respect to the stick fixed neutral point. Subtracting equation (8.1.7) from equation (8.2.7) gives

$$\frac{x_{n_{free}} - x_{n_{fix}}}{\bar{c}} = -\frac{C_{N_{h_\delta}}}{C_{N_\alpha}} \frac{C_{h_\alpha}}{C_{h_\delta}} \left(1 - \frac{d\varepsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} = \frac{C_{m_\delta}}{C_{N_\alpha}} \frac{C_{h_\alpha}}{C_{h_\delta}} \left(1 - \frac{d\varepsilon}{d\alpha}\right). \tag{8.2.9}$$

8.2.4 Elevator stick forces

Now we will examine the stick forces which the pilot should exert. We denote the **stick deflection** by s_e. By considering the work done by the pilot, we find that $F_e ds_e + H_e d\delta_e = 0$. From this follows that the **stick force** F_e is given by

$$F_e = -\frac{d\delta_e}{ds_e} H_e = -\frac{d\delta_e}{ds_e} C_{h_e} \frac{1}{2} \rho V_h^2 S_e \bar{c}_e. \tag{8.2.10}$$

By the way S_e is the **elevator surface** and $\bar{c}_e$ is the **mean elevator chord**. If we massively rewrite the above equation, we can eventually find that

$$F_e = -\frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left(C'_{h_0} \frac{1}{2} \rho V^2 + C'_{h_\alpha} \frac{W}{S} \frac{1}{C_{N_\alpha}}\right). \tag{8.2.11}$$

We see that F_e consists of two parts. One part varies with the airspeed, while the other part does not. By the way, the coefficients C'_{h_0} and C'_{h_α} are given by

$$C'_{h_0} = -\frac{C_{h_\delta}}{C_{m_\delta}} C_{m_{ac}} - \frac{C_{h_\delta}}{C_{N_{h_\delta}}} C_{N_{h_{\alpha_{free}}}} (\alpha_0 + i_h) + C_{h_{\delta_t}} \delta_{t_e}, \tag{8.2.12}$$

$$C'_{h_\alpha} = -\frac{C_{h_\delta}}{C_{m_\delta}} C_{m_{\alpha_{free}}} = -\frac{C_{h_\delta}}{C_{m_\delta}} C_{N_\alpha} \frac{x_{cg} - x_{n_{free}}}{\bar{c}}. \tag{8.2.13}$$

We see that C'_{h_0} depends on δ_{t_e}. To simplify our equation, we can apply a small trick. We define $\delta_{t_{e_0}}$ to be the value of δ_{t_e} for which $C'_{h_0} = 0$. It follows that

$$\delta_{t_{e_0}} = \frac{1}{C_{h_{\delta_t}}} \left(\frac{C_{h_\delta}}{C_{m_\delta}} C_{m_{ac}} + \frac{C_{h_\delta}}{C_{N_{h_\delta}}} C_{N_{h_{\alpha_{free}}}} (\alpha_0 + i_h)\right). \tag{8.2.14}$$

We can now rewrite the stick deflection force as

$$F_e = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left(\frac{W}{S} \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \frac{x_{cg} - x_{n_{free}}}{\bar{c}} - \frac{1}{2} \rho V^2 C_{h_{\delta_t}} \left(\delta_{t_e} - \delta_{t_{e_0}}\right)\right). \tag{8.2.15}$$

The control forces, which the pilots need to exert, greatly determine how easy and comfortable it is to fly an airplane. The above equation is therefore rather important.

We can also derive something else from the above equation. Let's define the **trim speed** V_{tr} to be the speed at which $F_e = 0$. We now examine the derivative dF_e/dV at this trim speed. (So at $F_e = 0$.) If it is positive ($dF_e/dV > 0$), then the aircraft is said to have **elevator control force stability** in the current flight condition. It can be shown that this derivative is given by

$$\left(\frac{dF_e}{dV}\right)_{F_e=0} = -2 \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V_{tr}}\right)^2 \frac{W}{S} \frac{C_{h_\delta}}{C_{m_{\delta_e}}} \frac{x_{cg} - x_{n_{free}}}{\bar{c}} \frac{1}{V_{tr}}. \tag{8.2.16}$$

It's the job of the designer to keep this derivative positive.

8.3 Longitudinal control

8.3.1 Special manoeuvres

Previously, we have only considered steady flight. Now we suppose that we are performing some special manoeuvre. We will consider both a steady pull-up manoeuvre and a horizontal steady turn.

During these manoeuvres, we will have a certain **load factor** $n = N/W$. There are two parameters that are important for the manoeuvres. They are the **elevator deflection per** g, denoted by $d\delta_e/dn$, and the **stick force per** g, denoted by dF_e/dn. Both these parameters should be negative. And they may not be too high or too low either.

8.3.2 The elevator deflection per g

We will now find an expression for $d\delta_e/dn$. Let's suppose we're initially in a horizontal steady flight. But after a brief moment, we'll be in one of the special manoeuvres. In this brief moment, several aircraft parameters have changed.

Let's examine the change in normal force ΔC_N and the change in moment ΔC_m. The change in normal force is effected by the angle of attack α and the pitch rate q. This gives us

$$\Delta C_N = \frac{\Delta N}{\frac{1}{2}\rho V^2 S} = \frac{W}{\frac{1}{2}\rho V^2 S}\Delta n = C_{N_\alpha}\Delta\alpha - C_{Z_q}\Delta\frac{q\bar{c}}{V}. \tag{8.3.1}$$

Similarly, the change in moment is effected by the angle of attack α, the pitch rate q and the elevator deflection δ_e. This gives us

$$\Delta C_m = 0 = C_{m_\alpha}\Delta\alpha + C_{m_q}\Delta\frac{q\bar{c}}{V} + C_{m_{\delta_e}}\Delta\delta_e. \tag{8.3.2}$$

You may wonder, why is $\Delta C_m = 0$? This is because both in the initial situation and the final situation, we have a steady manoeuvre. There is thus no angular acceleration present. The moment must thus stay constant.

From the first of the above two equations, we can find the derivative of α with respect to n. It is given by

$$\frac{d\alpha}{dn} = \frac{1}{C_{N_\alpha}}\frac{W}{\frac{1}{2}\rho V^2 S} + \frac{C_{Z_q}}{C_{N_\alpha}}\frac{d\frac{q\bar{c}}{V}}{dn}. \tag{8.3.3}$$

From the second of these equations, we can find that

$$\frac{d\delta_e}{dn} = -\frac{1}{C_{m_{\delta_e}}}\left(C_{m_\alpha}\frac{d\alpha}{dn} + C_{m_q}\frac{d\frac{q\bar{c}}{V}}{dn}\right). \tag{8.3.4}$$

Inserting the value of $d\alpha/dn$ will eventually give us

$$\frac{d\delta_e}{dn} = -\frac{1}{C_{m_{\delta_e}}}\left(\frac{C_{m_\alpha}}{C_{N_\alpha}}\frac{W}{\frac{1}{2}\rho V^2 S} + \left(\frac{C_{m_\alpha}C_{Z_q}}{C_{N_\alpha}} + C_{m_q}\right)\frac{d\frac{q\bar{c}}{V}}{dn}\right). \tag{8.3.5}$$

We will determine the term $d\frac{q\bar{c}}{V}/dn$ later, since it depends on the type of manoeuvre that is being performed.

8.3.3 The stick force per g

It's time to find an expression for dF_e/dn. From equation (8.2.10), we can derive that

$$\frac{dF_e}{dn} = -\frac{d\delta_e}{ds_e}\frac{1}{2}\rho V_h^2 S_e\bar{c}_e\left(C_{h_\alpha}\frac{d\alpha_h}{dn} + C_{h_\delta}\frac{d\delta_e}{dn}\right). \tag{8.3.6}$$

We already have an expression for $d\delta_e/dn$. The expression for α_h is a bit tricky. This is because we also have a rotation q. If we take this into account, we will have

$$\alpha_h = (\alpha - \alpha_0)\left(1 - \frac{d\varepsilon}{d\alpha}\right) + (\alpha_0 + i_h) + \frac{l_h}{\bar{c}}\frac{q\bar{c}}{V}. \tag{8.3.7}$$

The derivative of α_h, with respect to n, will then be

$$\frac{d\alpha_h}{dn} = \left(1 - \frac{d\varepsilon}{d\alpha}\right)\frac{d\alpha}{dn} + \frac{l_h}{\bar{c}}\frac{d\frac{q\bar{c}}{V}}{dn}. \tag{8.3.8}$$

Luckily, we still remember $d\alpha/dn$ from equation (8.3.3). From this, we can derive an equation that's way too long to write down here. However, once we examine specific manoeuvres, we will mention the final equation.

8.3.4 The pull-up manoeuvre

Let's consider an aircraft in a pull-up manoeuvre. When an aircraft pulls its nose up, the pilot will experience higher g-forces. This will thus cause the load factor n to change.

To be able to study pull-up manoeuvres, we simplify them. We assume that both n and V are constant. If this is the case, the aircraft's path will form a part of a circle. The centripetal accelaration thus is $N - W = mVq$. By using $n = N/W$ and $W = mg$, we can rewrite this as

$$\frac{q\bar{c}}{V} = \frac{g\bar{c}}{V^2}(n-1). \tag{8.3.9}$$

Differentiating with respect to n gives

$$\frac{d\frac{q\bar{c}}{V}}{dn} = \frac{g\bar{c}}{V^2} = \frac{1}{2\mu_c}\frac{W}{\frac{1}{2}\rho V^2 S}, \qquad \text{where} \quad \mu_c = \frac{m}{\rho S\bar{c}} = \frac{W}{g\rho S\bar{c}}. \tag{8.3.10}$$

By using this, we can find the elevator deflection per g for a pull-up manoeuvre. It is

$$\frac{d\delta_e}{dn} = -\frac{1}{C_{m_{\delta_e}}}\frac{W}{\frac{1}{2}\rho V^2 S}\left(\frac{C_{m_\alpha}}{C_{N_\alpha}}\left(1 + \frac{C_{Z_q}}{2\mu_c}\right) + \frac{C_{m_q}}{2\mu_c}\right). \tag{8.3.11}$$

Often the term $C_{Z_q}/2\mu_c$ can be neglected. This simplifies matters a bit. We can also derive a new expression for the stick force per g. We will find that

$$\frac{dF_e}{dn} = \frac{d\delta_e}{ds_e}\frac{W}{S}\left(\frac{V_h}{V}\right)^2 S_e\bar{c}_e\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\left(\frac{C_{m_{\alpha_{free}}}}{C_{N_\alpha}} + \frac{C_{m_{q_{free}}}}{2\mu_c}\right). \tag{8.3.12}$$

In this equation, we can see the parameters $C_{m_{\alpha_{free}}}$ and $C_{m_{q_{free}}}$. These are the values of C_{m_α} and C_{m_q} when the pilot releases his stick. They are given by

$$C_{m_{\alpha_{free}}} = C_{N_{w_\alpha}}\frac{x_{cg} - x_w}{\bar{c}} + C_{m_{\alpha_{h_{free}}}} \qquad \text{and} \qquad C_{m_{q_{free}}} = C_{m_q} - C_{m_{\delta_e}}\frac{C_{h_\alpha}}{C_{N_\alpha}}\frac{l_h}{\bar{c}}. \tag{8.3.13}$$

(The relation for $C_{m_{\alpha_{h_{free}}}}$ was already given in equation (8.2.5).)

8.3.5 The steady horizontal turn

Now let's consider an aircraft in a steady horizontal turn. It is performing this turn with a constant roll angle φ. From this, we can derive that

$$N\cos\varphi = W \qquad \text{and} \qquad N - W\cos\varphi = mVq. \tag{8.3.14}$$

If we combine the above relations, and rewrite them, we will get

$$\frac{q\bar{c}}{V} = \frac{g\bar{c}}{V^2}\left(n - \frac{1}{n}\right). \tag{8.3.15}$$

Differentiating with respect to n will then give us

$$\frac{d\frac{q\bar{c}}{V}}{dn} = \frac{1}{2\mu_c}\frac{W}{\frac{1}{2}\rho V^2 S}\left(1 + \frac{1}{n^2}\right). \tag{8.3.16}$$

By using this, we can find the elevator deflection per g for a horizontal steady turn. It is

$$\frac{d\delta_e}{dn} = -\frac{1}{C_{m_{\delta_e}}}\frac{W}{\frac{1}{2}\rho V^2 S}\left(\frac{C_{m_\alpha}}{C_{N_\alpha}} + \left(\frac{C_{m_\alpha}}{C_{N_\alpha}}\frac{C_{Z_q}}{2\mu_c} + \frac{C_{m_q}}{2\mu_c}\right)\left(1 + \frac{1}{n^2}\right)\right). \tag{8.3.17}$$

Again, we may often assume that $C_{Z_q}/2\mu_c \approx 0$. This again simplifies the equation. We also have the stick force per g. In this case, it is given by

$$\frac{dF_e}{dn} = \frac{d\delta_e}{ds_e}\frac{W}{S}\left(\frac{V_h}{V}\right)^2 S_e\bar{c}_e\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\left(\frac{C_{m_{\alpha_{free}}}}{C_{N_\alpha}} + \frac{C_{m_{q_{free}}}}{2\mu_c}\left(1 + \frac{1}{n^2}\right)\right). \tag{8.3.18}$$

It is interesting to see the similarities between the pull-up manoeuvre and the steady horizontal turn. In fact, if the load factor n becomes big, the difference between the two manoeuvres disappears.

8.3.6 The manoeuvre point

An important point on the aircraft, when performing manoeuvres, is the **manoeuvre point**. It is defined as the position of the CG for which $d\delta_e/dn = 0$. First we will examine the **stick fixed manoeuvre point** $x_{m_{fix}}$. To have $d\delta_e/dn = 0$ for a pull-up manoeuvre (neglecting $C_{Z_q}/2\mu_c$), we should have

$$\frac{C_{m_\alpha}}{C_{N_\alpha}} + \frac{C_{m_q}}{2\mu_c} = \frac{x_{cg} - x_{n_{fix}}}{\bar{c}} + \frac{C_{m_q}}{2\mu_c} = 0. \tag{8.3.19}$$

If the above equation holds, then the CG equals the manoeuvre point. We thus have

$$\frac{x_{m_{fix}} - x_{n_{fix}}}{\bar{c}} = -\frac{C_{m_q}}{2\mu_c} \qquad \text{and also} \qquad \frac{x_{cg} - x_{m_{fix}}}{\bar{c}} = \frac{C_{m_\alpha}}{C_{N_\alpha}} + \frac{C_{m_q}}{2\mu_c}. \tag{8.3.20}$$

(Remember that the above equations are for the pull-up manoeuvre. For the steady turn, we need to multiply the term with C_{m_q} by an additional factor $\left(1 + 1/n^2\right)$.) By using the above results, we can eventually obtain that

$$\frac{d\delta_e}{dn} = -\frac{1}{C_{m_{\delta_e}}}\frac{W}{\frac{1}{2}\rho V^2 S}\frac{x_{cg} - x_{m_{fix}}}{\bar{c}}. \tag{8.3.21}$$

By the way, this last equation is valid for both the pull-up manoeuvre and the steady horizontal turn.

We can also find the **stick free manoeuvre point** $x_{m_{free}}$. This goes, in fact, in a rather similar way. We will thus also find, for the pull-up manoeuvre, that

$$\frac{x_{m_{free}} - x_{n_{free}}}{\bar{c}} = -\frac{C_{m_{q_{free}}}}{2\mu_c} \qquad \text{and} \qquad \frac{x_{cg} - x_{m_{free}}}{\bar{c}} = \frac{C_{m_{\alpha_{free}}}}{C_{N_\alpha}} + \frac{C_{m_{q_{free}}}}{2\mu_c}. \tag{8.3.22}$$

(For the steady turn, we again need to multiply the term with $C_{m_{q_{free}}}$ by $\left(1 + 1/n^2\right)$.)

9. Lateral stability and control

In this chapter, we will examine lateral stability and control. How should we control an aircraft in a non-symmetrical steady flight?

9.1 The equations of motion

9.1.1 Derivation of the equations of motion for asymmetric flight

Let's examine an aircraft in a steady asymmetric flight. It has a roll angle φ and a sideslip angle β. By examining equilibrium, we can find that

$$W \sin\varphi + Y = mVr, \qquad L = 0 \quad \text{and} \quad N = 0. \tag{9.1.1}$$

Non-dimensionalizing these equations gives

$$C_L \varphi - 4\mu_b \frac{rb}{2V} + C_Y = 0, \qquad C_l = 0 \quad \text{and} \quad C_n = 0, \tag{9.1.2}$$

where we have $\mu_b = \frac{m}{\rho S b}$. We can also apply linearization to the above equations. This will then give us

$$C_L \varphi + C_{Y_\beta}\beta + (C_{Y_r} - 4\mu_b)\frac{rb}{2V} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r = 0, \tag{9.1.3}$$

$$C_{l_\beta}\beta + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r = 0, \tag{9.1.4}$$

$$C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = 0. \tag{9.1.5}$$

9.1.2 Simplifying the equations of motion

Let's examine the equations of the previous paragraph. There are quite some terms in these equations that are negligible. They are $C_{Y_{\delta_a}}$, $C_{l_{\delta_r}}$, C_{Y_r}, $C_{n_{\delta_a}}$ and $C_{Y_{\delta_r}}$. By using these neglections, and by putting the above equations into matrix form, we will get

$$\begin{bmatrix} C_L & C_{Y_\beta} & -4\mu_b & 0 & 0 \\ 0 & C_{l_\beta} & C_{l_r} & C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & C_{n_r} & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \varphi \\ \beta \\ \frac{rb}{2V} \\ \delta_a \\ \delta_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9.1.6}$$

Let's assume that the velocity V is already set. We then still have five unknowns and three equations. That means that there are infinitely many solutions. This makes sense: You can make a turn in infinitely many ways. How do we deal with this? We simply set one of the parameters. We then express three of the remaining parameters as a function of the last parameter. (This fifth parameter is usually $\frac{rb}{2V}$.) So let's do that.

9.2 Steady horizontal turns

9.2.1 Turns using ailerons only

Let's try to turn the aircraft, by only using ailerons. We do not use the rudder and thus have $\delta_r = 0$. We can insert this into the equations of motion. We then solve for the parameters β, φ and δ_a. This will give us

$$\frac{d\beta}{d\frac{rb}{2V}} = -\frac{C_{n_r}}{C_{n_\beta}} > 0 \qquad (\text{since } C_{n_r} < 0 \text{ and } C_{n_\beta} > 0), \tag{9.2.1}$$

$$\frac{d\varphi}{d\frac{rb}{2V}} = \frac{4\mu_b + C_{Y_\beta}\frac{C_{n_r}}{C_{n_\beta}}}{C_L} > 0, \tag{9.2.2}$$

$$\frac{d\delta_a}{d\frac{rb}{2V}} = \frac{1}{C_{l_{\delta_a}}}\frac{C_{l_\beta}C_{n_r} - C_{l_r}C_{n_\beta}}{C_{n_\beta}}. \tag{9.2.3}$$

The sign of the last equation is still a point of discussion. We would like to have $d\delta_a/d\frac{rb}{2V}$. If this is the case, then we have so-called **spiral stability**. We know that $C_{l_{\delta_a}} < 0$ and $C_{n_\beta} > 0$. So spiral stability is achieved if

$$C_{l_\beta}C_{n_r} - C_{l_r}C_{n_\beta} > 0. \tag{9.2.4}$$

We will find out in the next chapter why they call this the spiral stability condition.

9.2.2 Turns using the rudder only

We can also make a turn using only the rudder. So we have $\delta_a = 0$. This again gives us three equations, being

$$\frac{d\beta}{d\frac{rb}{2V}} = -\frac{C_{l_r}}{C_{l_\beta}} > 0 \qquad (\text{since } C_{l_r} > 0 \text{ and } C_{l_\beta} < 0), \tag{9.2.5}$$

$$\frac{d\varphi}{d\frac{rb}{2V}} = \frac{4\mu_b + C_{Y_\beta}\frac{C_{l_r}}{C_{l_\beta}}}{C_L} > 0, \tag{9.2.6}$$

$$\frac{d\delta_r}{d\frac{rb}{2V}} = -\frac{1}{C_{n_{\delta_r}}}\frac{C_{l_\beta}C_{n_r} - C_{l_r}C_{n_\beta}}{C_{l_\beta}}. \tag{9.2.7}$$

In the last equation, we have $C_{n_{\delta_r}} < 0$ and $C_{l_\beta} < 0$. If there is also spiral stability, then we have $d\delta_r/d\frac{rb}{2V} < 0$.

9.2.3 Coordinated turns

In a **coordinated turn**, we have $\beta = 0$. This means that there is no sideward component of the force acting on the aircraft. This is an important factor for passenger comfort. For the coordinated turn, we again have three equations. They are

$$\frac{d\varphi}{d\frac{rb}{2V}} = \frac{4\mu_b}{C_L} > 0, \tag{9.2.8}$$

$$\frac{d\delta_a}{d\frac{rb}{2V}} = -\frac{C_{l_r}}{C_{l_{\delta_a}}} > 0 \qquad (\text{since } C_{l_r} > 0 \text{ and } C_{l_{\delta_a}} < 0), \tag{9.2.9}$$

$$\frac{d\delta_r}{d\frac{rb}{2V}} = -\frac{C_{n_r}}{C_{n_{\delta_r}}} < 0 \qquad (\text{since } C_{n_r} < 0 \text{ and } C_{n_{\delta_r}} < 0). \tag{9.2.10}$$

9.2.4 Flat turns

If we want the aircraft to stay flat during the turns, then we have $\varphi = 0$. It then follows that

$$\frac{d\beta}{d\frac{rb}{2V}} = \frac{4\mu_b}{C_{Y_\beta}} < 0. \tag{9.2.11}$$

From this, we can also derive that

$$\frac{d\delta_a}{d\frac{rb}{2V}} > 0 \qquad \text{and} \qquad \frac{d\delta_r}{d\frac{rb}{2V}} < 0. \tag{9.2.12}$$

9.3 Other flight types

9.3.1 Steady straight sideslipping flight

Let's examine a steady straight sideslipping flight. This type of flight is usually only used during landings with strong sidewinds. However, sometimes the aircraft is brought into a steady straight sideslipping flight involuntarily. It is therefore important to know how the aircraft behaves.

In a straight flight, we have $\frac{rb}{2V} = 0$. We can now derive that

$$\frac{d\varphi}{d\beta} = -\frac{C_{Y_\beta}}{C_L} > 0, \qquad \frac{d\delta_a}{d\beta} = -\frac{C_{l_\beta}}{C_{l_{\delta_a}}} \qquad \text{and} \qquad \frac{d\delta_r}{d\beta} = -\frac{C_{n_\beta}}{C_{n_{\delta_r}}}. \tag{9.3.1}$$

We generally want to have $d\delta_a/d\beta < 0$ and $d\delta_r/d\beta > 0$. We also always have $C_{l_{\delta_a}} < 0$ and $C_{n_{\delta_r}} < 0$. This implies that we should have $C_{l_\beta} < 0$ and $C_{n_\beta} > 0$.

9.3.2 Stationary flight with asymmetric power

Let's suppose one of the engines of the aircraft doesn't work anymore. In this case, a yawing moment will be present. This moment has magnitude

$$C_{n_e} = k\frac{\Delta T_p y_e}{\frac{1}{2}\rho V^2 S b}. \tag{9.3.2}$$

The variable ΔT_p consists of two parts. First there is the reduction in thrust. Then there is also the increase in drag of the malfunctioning engine. y_e is the Y coordinate of the malfunctioning engine. Finally, k is an additional parameter, taking into account other effects. Its value is usually between 1.5 and 2.

Now let's try to find a way in which we can still perform a steady straight flight. (We should thus have $r = 0$.) We now have four unknowns and three equations. So we can still set one parameter. Usually, we would like to have $\varphi = 0$ as well. In this case, a sideslip angle β is unavoidable. If the right engine is inoperable, then a positive rudder deflection and sideslip angle will be present.

We could also choose to have $\beta = 0$. In this case, we will constantly fly with a roll angle φ. The wing with the inoperable engine then has to be lower than the other wing. So if the right wing malfunctions, then we have a positive roll angle.

10. Aircraft modes of vibration

It is finally time to look at the dynamics of an aircraft. How will an aircraft behave, when given elevator, rudder and aileron deflections? What are its modes of vibration? That's what we will look at in this chapter.

10.1 Eigenvalue theory

10.1.1 Solving the system of equations

To examine the dynamic stability of the aircraft, we examine the full longitudinal equations of motion. The symmetric part of these equations were

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) D_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{10.1.1}$$

In this system of equations, we have assumed stick-fixed conditions. All inputs are zero. We could try to find solutions for the above system of equations. A common way to do this, is to assume a solution of the form

$$\mathbf{x}(t) = \mathbf{A} e^{\lambda_c s_c}. \tag{10.1.2}$$

In this equation, $\mathbf{x}(t)$ is our solution. $s_c = \frac{V}{\bar{c}} t$ is the dimensionless time. From this form follows that $D_c \mathbf{x} = \lambda_c \mathbf{x}$. If we insert this into the equations of motion, we find

$$\begin{bmatrix} C_{X_u} - 2\mu_c \lambda_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) \lambda_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -\lambda_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} \lambda_c & 0 & C_{m_q} - 2\mu_c K_Y^2 \lambda_c \end{bmatrix} \begin{bmatrix} A_u \\ A_\alpha \\ A_\theta \\ A_q \end{bmatrix} e^{\lambda_c s_c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{10.1.3}$$

We can write this matrix equation as $[\Delta]\, \mathbf{A} e^{\lambda_c s_c} = \mathbf{0}$. The exponential in this equation can't be zero, so we can get rid of it. We thus need to solve $[\Delta]\, \mathbf{A} = \mathbf{0}$. One solution of this equation is $\mathbf{A} = 0$. However, this is a rather trivial solution, in which we are not interested. So we need to find non-trivial solutions. This is where our knowledge on linear algebra comes in. There can only be non-trivial solutions, if $\det[\Delta] = 0$. Applying this will give us an equation of the form

$$A\lambda_c^4 + B\lambda_c^3 + C\lambda_c^2 + D\lambda_c + E = 0. \tag{10.1.4}$$

This equation is called the **characteristic polynomial**. Solving it will give four **eigenvalues** λ_{c_1}, λ_{c_2}, λ_{c_3} and λ_{c_4}. Corresponding to these four eigenvalues are four **eigenvectors** $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ and $\mathbf{A_4}$. The final solution of the system of equations now is

$$\mathbf{x} = c_1 \mathbf{A_1} e^{\lambda_{c_1} s_c} + c_2 \mathbf{A_2} e^{\lambda_{c_2} s_c} + c_3 \mathbf{A_3} e^{\lambda_{c_3} s_c} + c_4 \mathbf{A_4} e^{\lambda_{c_4} s_c}. \tag{10.1.5}$$

The constants c_1, c_2, c_3 and c_4 depend on the four initial conditions.

10.1.2 The eigenvalues

Let's examine the eigenvalues of the system of equations. Each eigenvalue can be either real and complex. If one of the eigenvalues is complex, then its complex conjugate is also an eigenvalue. Complex eigenvalues therefore always come in pairs.

A **mode of vibration** is a characteristic way in which an object (our aircraft) can vibrate. The number of modes depends on the eigenvalues. In fact, it is equal to the number of different eigenvalues. (When performing the counting, a pair of complex conjugate eigenvalues is counted as one.) For example, an aircraft with two real eigenvalues and two complex eigenvalues has three modes of vibration.

The eigenvalues λ_c are very important for the stability of the system. To examine stability, we look at the limit

$$\lim_{s_c \to \infty} c_1 \mathbf{A_1} e^{\lambda_{c_1} s_c} + c_2 \mathbf{A_2} e^{\lambda_{c_2} s_c} + c_3 \mathbf{A_3} e^{\lambda_{c_3} s_c} + c_4 \mathbf{A_4} e^{\lambda_{c_4} s_c}. \tag{10.1.6}$$

If only one of the eigenvalues has a positive real part, then this limit will diverge. This means that our aircraft is **unstable**. If, however, all eigenvalues have negative real parts, then the system is **stable**.

10.1.3 Real eigenvalue properties

We can derive some interesting properties from the eigenvalues. First, let's examine a real eigenvalue λ_c. This eigenvalue has its own mode of vibration $\mathbf{x} = \mathbf{A}e^{\lambda_c s_c}$. The **half time** $T_{\frac{1}{2}}$ is defined as the time it takes to reduce the amplitude of the motion to half of its former magnitude. In other words, $\mathbf{x}(t + T_{\frac{1}{2}}) = \frac{1}{2}\mathbf{x}(t)$. Solving this equation will give

$$T_{\frac{1}{2}} = \frac{\ln \frac{1}{2}}{\lambda_c} \frac{\bar{c}}{V}. \tag{10.1.7}$$

Similarly, we define the time constant τ as the time it takes for the amplitude to become $1/e$ of its former magnitude. Solving $\mathbf{x}(t + \tau) = \frac{1}{e}\mathbf{x}(t)$ gives

$$\tau = -\frac{1}{\lambda_c} \frac{\bar{c}}{V}. \tag{10.1.8}$$

These two parameters of course only exist if λ_c is negative. If it is positive, then the magnitude will only grow. In this case, the **doubling time** T_2 is an important parameter. It is given by $T_2 = -T_{\frac{1}{2}}$.

10.1.4 Complex eigenvalue properties

Now let's examine a complex eigenvalue pair. We can write it as $\lambda_{c_{1,2}} = \xi_c \pm \eta_c i$, where $i = \sqrt{-1}$ is the complex number. This eigenvalue will cause an oscillation. The period and frequency of the oscillation only depend on η_c. In fact, the **period** P, the **frequency** f and the **angular frequency** ω_n are given by

$$P = \frac{2\pi}{\eta_c} \frac{\bar{c}}{V}, \qquad f = \frac{1}{P} = \frac{\eta_c}{2\pi} \frac{V}{\bar{c}} \qquad \text{and} \qquad \omega_n = \frac{2\pi}{P} = \eta_c \frac{V}{\bar{c}}. \tag{10.1.9}$$

The damping of this oscillation is caused by the real part ξ_c. Again, the **half time** $T_{\frac{1}{2}}$ is defined as the time it takes for the amplitude to reduce to half its size. It is still given by

$$T_{\frac{1}{2}} = \frac{\ln \frac{1}{2}}{\xi_c} \frac{\bar{c}}{V}. \tag{10.1.10}$$

Another important parameter is the **logarithmic decrement** δ. It is defined as the natural logarithm of the ratio of the magnitude of two successive peaks. In other words, it is defined as

$$\delta = \ln \left(\frac{e^{\xi_c \frac{V}{\bar{c}}(t+P)}}{e^{\xi_c \frac{V}{\bar{c}} t}} \right) = \xi_c \frac{V}{\bar{c}} P. \tag{10.1.11}$$

Finally, there are the **damping ratio** ζ and the **undamped angular frequency** ω_0. They are defined such that

$$\lambda_{c_{1,2}} = \left(-\zeta\omega_0 \pm i\omega_0\sqrt{1 - \zeta^2} \right) \frac{\bar{c}}{V}. \tag{10.1.12}$$

Solving for ζ and ω_0 will give

$$\zeta = \frac{-\xi_c}{\sqrt{\xi_c^2 + \eta_c^2}} \qquad \text{and} \qquad \omega_0 = \sqrt{\xi_c^2 + \eta_c^2}\frac{V}{\bar{c}}. \tag{10.1.13}$$

10.1.5 Getting stable eigenvalues

Let's take a look at the characteristic equation (equation (10.1.4)). We usually set up the equation, such that $A > 0$. To obtain four eigenvalues with negative real parts, we must have

$$B > 0, \quad C > 0, \quad D > 0 \quad \text{and} \quad E > 0. \tag{10.1.14}$$

But these aren't the only conditions to ensure that we have stable eigenvalues. We must also have

$$R = BCD - AD^2 - B^2E > 0. \tag{10.1.15}$$

These criteria are known as the **Routh-Hurwitz Stability Criteria**. The coefficient R is called **Routh's discriminant**. These criteria hold for both the symmetric and the asymmetric modes of vibration.

10.2 The symmetric modes of vibration

10.2.1 Example eigenvalues

Let's suppose we know all the parameters in the matrix equation that was described earlier. In this case, we can find the four eigenvalues. An example solution of these eigenvalues is given by

$$\lambda_{c_{1,2}} = -0.04 \pm 0.04i \qquad \text{and} \qquad \lambda_{c_{3,4}} = -0.0003 \pm 0.006i. \tag{10.2.1}$$

Of course these values will be different for different aircraft. But most types of aircraft, having the standard wing-fuselage-tailplane set-up, will have similar eigenvalues.

Let's study these eigenvalues. There are two pairs of complex conjugate eigenvalues. Both pairs of eigenvalues have negative real parts. This means that the aircraft is stable. Since there are only two pairs of complex conjugate eigenvalues, there are two modes of vibration. We will now examine these modes.

10.2.2 The short period oscillation

Let's look at the first pair of eigenvalues. It has a relatively big real part ξ_c The damping is therefore big. The complex part η_c is relatively big as well. So the frequency is high. In other words, we have a highly damped high-frequency oscillation. This motion is known as the **short period oscillation**.

Let's take a look at what actually happens in the aircraft. We start the short period oscillation by applying a step input to the elevator deflection. (We deflect it, and keep that deflection.) We can, for example, deflect it upward. This causes the lift on the horizontal tailplane to decrease. This, in turn, causes the pitch rate to increase. An increase in pitch rate will, however, increase the effective angle of attack of the horizontal tailplane. This then reduces the pitch rate. And the whole cycle starts over again. However, the oscillation is highly damped. After less than one period, the effects are hardly noticable anymore.

Now let's try to derive some equations for the short period motion. The short period motion is rather fast. So we assume the aircraft hasn't had time yet to change its velocity in X or Z direction. This means that $\hat{u} = 0$ and $\gamma = 0$. Therefore $\alpha = \theta$. This reduces the equations of motion to

$$\begin{bmatrix} C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c)\lambda_c & 2\mu_c + C_{Z_q} \\ C_{m_\alpha} + C_{m_{\dot{\alpha}}}\lambda_c & C_{m_q} - 2\mu_c K_Y^2 \lambda_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10.2.2}$$

We can now find the eigenvalues for this matrix. This will still give us a rather complicated equation. If we neglect $C_{Z_{\dot{\alpha}}}$ and C_{Z_q}, then this complicated equation reduces to

$$\lambda_{c_{1,2}} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{10.2.3}$$

In this equation, the coefficients A, B and C are given by

$$A = 4\mu_c^2 K_Y^2, \qquad B = -2\mu_c (K_Y^2 C_{Z_\alpha} + C_{m_{\dot{\alpha}}} + C_{m_q}) \quad \text{and} \quad C = C_{Z_\alpha} C_{m_q} - 2\mu_c C_{m_\alpha}. \tag{10.2.4}$$

To have stability, we should have $-\frac{B}{2A}$ negative. We know that A is positive. This means that B has to be positive as well.

10.2.3 The phugoid

Now we look at the second pair of eigenvalues. It has a small real part ξ_c, and therefore a small damping. The complex part η_c is small as well, so the frequency is low. In other words, we have a lightly damped low-frequency oscillation. This motion is known as the **phugoid**.

Again, we look at what happens with the aircraft. This time, we apply an impulse deflection on the elevator. (We only deflect it briefly.) This will cause our pitch angle to increase. (That is, after the short period motion has more or less damped out.) We will therefore go upward. This causes our velocity to decrease. Because of this, the lift is reduced. Slowly, the pitch angle will decrease again, and we will go downward. This causes the velocity to increase. This, in turn, increases the lift. The pitch angle will again increase, and we will again go upward.

Again, we will try to derive some relations for the phugoid. In the phugoid, the angle of attack α is approximately constant. (γ and θ do vary a lot though.) So we have $\alpha = 0$ and $\dot{\alpha} = 0$. (Remember that we're discussing deviations from the initial position.) Since the oscillation is very slow, we also assume that $\dot{q} = 0$. If we also neglect the terms C_{Z_q} and C_{X_0}, we will find that we again have

$$\lambda_{c_{3,4}} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{10.2.5}$$

However, now the coefficients are given by

$$A = -4\mu_c^2, \qquad B = 2\mu_c C_{X_u} \quad \text{and} \quad C = -C_{Z_u} C_{Z_0}. \tag{10.2.6}$$

We can apply the approximations $C_{X_u} = -2C_D$, $C_{Z_0} = -C_L$ and $C_{Z_u} = -2C_L$. This would then give us the three parameters

$$\omega_0 = \frac{V}{\bar{c}} \sqrt{\frac{C_L^2}{2\mu_c^2}} = \frac{g}{V}\sqrt{2}, \qquad \zeta = \frac{\sqrt{2}}{2}\frac{C_D}{C_L} \quad \text{and} \quad P = \frac{2\pi}{\omega_0\sqrt{1-\zeta^2}} \approx \frac{2\pi}{\omega_0} = \sqrt{2}\pi\frac{V}{g}. \tag{10.2.7}$$

Note that, in the above equation for P, we have used the fact that the damping ζ is small. Although the above equations are only approximations, they can serve as quite handy tools in verifying your results.

10.3 The asymmetric modes of vibration

10.3.1 Example eigenvalues

We have just examined the symmetric equations of motion. Of course, we can do the same for the asymmetric equations of motion. These equations of motion are

$$\begin{bmatrix} C_{Y_\beta} + (C_{Y_{\dot\beta}} - 2\mu_b)D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} + C_{n_{\dot\beta}} D_b & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{10.3.1}$$

Examining them goes in more or less the same way as for the symmetric case. There is, however, one important difference. Since we are examining the asymmetric case, we don't use the chord $\bar{c}$ but we use the wing span b. The eigenvalues are thus also denoted as λ_b. Example eigenvalues for an aircraft are

$$\lambda_{b_1} = -0.4, \qquad \lambda_{b_2} = 0.01 \qquad \text{and} \qquad \lambda_{b_{3,4}} = -0.04 \pm 0.4i. \tag{10.3.2}$$

You might be surprised that these eigenvalues are a lot bigger than the symmetric eigenvalues. This is not very important. It's only the case, because they are based on b, instead of on $\bar{c}$. And naturally, b is a lot bigger than $\bar{c}$.

Let's examine the eigenvalues. There are two real eigenvalues and one pair of complex conjugate eigenvalues. The aircraft thus has three modes of vibration. You might also have noticed that there is a positive eigenvalue. The aircraft is thus unstable. The eigenvalue is, however, very small. This means that the aircraft state will only diverge very slowly. The pilot will have plenty of time to do something about it. So you don't have to worry: flying is still safe.

10.3.2 The aperiodic roll

The motion corresponding to λ_{b_1} is called the **aperiodic roll**. The eigenvalue is very negative. This motion is therefore highly damped.

The aperiodic roll is induced by applying a step input to the aileron. When this happens, the aircraft will start rolling. Let's suppose it rolls to the right. The right wing then goes down. This means that the right wing will get a higher effective angle of attack. The lift of this wing thus increases. The opposite happens for the left wing: its lift decreases. This lift difference causes a moment opposite to the rolling motion. In other words, the motion is damped out. The roll rate p will converge rather quickly to a constant value.

The aperiodic roll is a very fast motion. So there is no time for sideslip or yaw effects to appear. So we can assume that, during an aperiodic roll motion, we have $\beta = r = 0$. This reduces the equations of motion to just one equation, being

$$\left(C_{l_p} - 4\mu_b K_X^2 D_b\right) \frac{pb}{2V} = 0. \tag{10.3.3}$$

It directly follows that the corresponding eigenvalue is given by

$$\lambda_{b_1} = \frac{C_{l_p}}{4\mu_b K_X^2}. \tag{10.3.4}$$

10.3.3 The spiral motion

The motion corresponding to λ_{b_2} is called the **spiral motion**. The eigenvalue is positive. So the motion is unstable. However, the eigenvalue is very small. This means that divergence will occur only very slowly. We thus say that the motion is **marginally unstable**. (For some aircraft, this value is slightly negative. Such aircraft are **marginally stable**.)

The spiral motion is induced by an initial roll angle. (An angle of 10° is sufficient.) This causes the lift vector to be tilted. The horizontal component of the lift will cause the aircraft to make a turn. In the meanwhile, the vertical component of the lift vector has slightly decreased. This causes the aircraft to lose altitude. Combining these two facts will mean that the aircraft will perform a spiral motion.

If the eigenvalue λ_{b_2} is positive, then the roll angle of the aircraft will slowly increase. The spiral motion will therefore get worse. After a couple of minutes, the roll angle might have increased to 50°. This phenomenon is, however, not dangerous. The pilot will have plenty of time to react. It is also very easy to pull the aircraft out of a spiral motion.

Let's try to derive an equation for λ_{b_2}. The spiral motion is a very slow motion. We thus neglect the derivatives of β, p and r. Also, the coefficients C_{Y_r} and C_{Y_p} are neglected. After working out some equations, we can eventually find that

$$\lambda_{b_2} = \frac{2C_L \left(C_{l_\beta} C_{n_r} - C_{n_\beta} C_{l_r}\right)}{C_{l_p} \left(C_{Y_\beta} C_{n_r} + 4\mu_b C_{n_\beta}\right) - C_{n_p} \left(C_{Y_\beta} C_{l_r} + 4\mu_b C_{l_\beta}\right)}. \tag{10.3.5}$$

The denominator of this relation is usually negative. We say we have **spiral stability** if $\lambda_{b_2} < 0$. This is thus the case if

$$C_{l_\beta} C_{n_r} - C_{n_\beta} C_{l_r} > 0. \tag{10.3.6}$$

You might remember that we've seen this equation before.

10.3.4 The Dutch roll

The pair of eigenvalues $\lambda_{b_{3,4}}$ has a slightly low damping and a slightly high frequency. In the mode of vibration corresponding to these eigenvalues, the aircraft alternately performs a yawing and a rolling motion. The mode of vibration is called the **Dutch roll**.

Let's take a look at what actually happens with the aircraft. To initiate the Dutch roll, an impulse input is applied to the rudder. This causes the aircraft to yaw. Let's suppose the aircraft yaws to the right. The lift on the left wing then increases, while the lift on the right wing decreases. This moment causes the aircraft to roll to the right.

When the aircraft is rolling to the right, then the lift vector of the right wing is tilted forward. Similarly, the left wing will have a lift vector that is tilted backward. This causes the aircraft to yaw to the left. (This effect is still called adverse yaw.) In this way, roll and yaw alternate each other. It is important to remember that roll and yaw are alternately present. When the roll rate is at a maximum, the yaw rate is approximately zero, and vice verse.

The Dutch roll is not very comfortable for passenger. To increase passenger comfort, a yaw damper is used. This is an automatic system, which uses rudder/aileron deflections to reduce the effects of the Dutch roll.

Let's try to find a relation for $\lambda_{b_{3,4}}$. This is rather hard, since both roll and yaw are present. However, experience has shown that we still get slightly accurate results, if we neglect the rolling part of the motion. We thus assume that $\varphi = p = 0$. This reduces the system of equations to a 2×2 matrix. From it, we can again find that

$$\lambda_{b_{3,4}} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{10.3.7}$$

However, this time the coefficients A, B and C are given by

$$A = 8\mu_b^2 K_Z^2, \qquad B = -2\mu_b \left(C_{n_r} + 2K_Z^2 C_{Y_\beta}\right) \qquad \text{and} \qquad C = 4\mu_b C_{n_\beta} + C_{Y_\beta} C_{n_r}. \tag{10.3.8}$$

And that concludes our discussion on the modes of vibration.

10.3.5 Stability criteria

From the characteristic equation (equarion (10.1.4)) we can see which eigenmotions are stable. We have seen earlier that, if A, B, C, D, E and R are all positive, then all eigenmotions are stable. In other words, we have spiral stability and a convergent Dutch roll.

However, if some of the coefficients become negative, then there will be unstable eigenmotions. If $E < 0$, then we have spiral instability. Similarly, if $R < 0$, then we will have a divergent Dutch roll. So, to ensure stability, we'd best keep the coefficient of the characteristic equation positive.